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Sufficient Conditions from Countably Lipschitz to be Strongly Generalized Absolutely Continuous

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Abstract The CL-integral is an integral that is defined by using countably Lipschitz condition. The CL-integral is more general than the Lebesgue integral, but it is more specific than Denjoy integral in the wide sense. The Denjoy integral in the wide sense is more general than the Henstock-Kurzweil integral. In this paper, it will be given some conditions such that a CL-integrable function is Henctock-Kurzweil-integrable on $[a, b]$.

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1. Introduction

Recently, the Henstock-Kurzweil integral is very famous. It is a non-absolute integral which is more general than the Lebesgue integral [5]. The primitive F of the Henstock-Kurzweil integrable function f on $[a, b]$ is continuous, strongly generalized absolutely continuous (ACG^*) on $[a, b]$, and $F' = f$ almost everywhere in $[a, b]$.

According to Bruckner [1], there exists an integral that is more general than the Lebesgue integral. Bruckner involves the concept of absolutely continuous in discussing his research. The ACG^* property is more general than absolutely continuous function and AC_δ. Starting in 2015, Indrati tried to weaken the concept of ACG^* to have a countably Lipschitz condition (CLC). In 2016, Indrati and Aryati introduced an integral based on countably Lipschitz condition which is called CL-integral. Using the globally Riemann sums (GSRS) property for the Henstock-Kurzweil integral, it has been proved that the CL-integral is more general than the Lebesgue integral [2]. The GSRS property is developed from locally small Riemann sums property [4, 6]. The CL-integral has been applied in Picard Theorem [3]. The CL-integral is more specific than the general Denjoy integral [2].

We recall some concepts that will be used in Main Results, including the definition of the Henstock-Kurzweil integral, the CL-integral, countably Lipschitz condition, and absolutely continuous as follows.

Definition 1.1. A function $f : [a, b] \to \mathbb{R}$ is said to be Henstock-Kurzweil integrable on $[a, b]$ with its HK-primitive F, if for every $\epsilon > 0$, there exists a positive function δ on $[a, b]$ such that for every δ-fine partition $\mathcal{D} = \{([u, v], x)\}$ of $[a, b]$ we have

$$(\mathcal{D})\sum |f(x)(v - u) - F(u, v)| < \epsilon,$$

where $F(u, v) = F(v) - F(u)$.

The symbol A^0 represents the set of all interior points of A. Two sets A and B, with $A^0 \cap B^0 = \emptyset$, are called non-overlapping.

Definition 1.2. Let $X \subseteq \mathbb{R}$ and $F : X \to \mathbb{R}$.

(i) The function F is said to satisfy countably Lipschitz condition (CLC) on X, in short $F \in CLC(X)$, if there exists a collection of sets $\{X_n\}$, such that $X = \cup X_n$ and F satisfies Lipschitz condition on X_n, for every n, i.e., for every n, there exists $M_n > 0$, such that for every $x, y \in X_n$, we have

$$|F(x) - F(y)| \leq M_n|x - y|.$$

(ii) The function F is said to be absolutely continuous (AC) on X, in short $F \in AC(X)$, if for every $\epsilon > 0$, there exists a positive δ such that for any finite or infinite collection of non-overlapping intervals $\{I_i = [a_i, b_i]\}$, $a_i, b_i \in X$, if $\sum |b_i - a_i| < \delta$ then

$$\left|\sum [F(b_i) - F(a_i)]\right| < \epsilon.$$

(iii) The function F is said to be strongly absolutely continuous (AC^*) on X, in short $F \in AC^*(X)$, if for every $\epsilon > 0$, there exists a positive δ such that for any finite or infinite collection of non-overlapping intervals $\{I_i = [a_i, b_i]\}$, $a_i, b_i \in X$, if $\sum |b_i - a_i| < \delta$ then

$$\left|\sum \omega(F; I_i)\right| < \epsilon,$$

where $\omega(F; I_i)$ stands for the oscillation of the function F on I_i.

(iv) The function F is said to be generalized absolutely continuous (ACG) on X, in short $F \in ACG(X)$, if there exists a collection of sets $\{X_n\}$, such that $X = \cup X_n$ and $F \in AC(X_n)$ for every n.

(v) The function F is said to be strongly generalized absolutely continuous (ACG^*) on X, in short $F \in ACG^*(X)$, if there exists a collection of sets $\{X_n\}$, such that $X = \cup X_n$ and $F \in AC^*(X_n)$ for every n.

Here, X_n may not an interval for some n. As corollary, if $x, y \in X_n$ with $x < y$, the interval $[x, y]$ may not subset of X_n.

Lemma 1.3. *Let $X \subseteq [a, b]$ and the function $F : [a, b] \to \mathbb{R}$ is continuous.*

(i) If $F \in AC(X)$, then $F \in AC(\overline{X})$.

(ii) If $F \in AC^*(X)$, then $F \in AC^*(\overline{X})$.

Based on Lemma 1.3, the collection of $\{X_n\}$ in the definition $ACG(X)$ and $ACG^*(X)$, the set X_n can be assumed a closed set for every n.

From the countably Lipschitz condition properties, it has been defined a CL-integral as stated in Definition 1.4.

Definition 1.4. A function $f : [a, b] \to \mathbb{R}$ is said to be CL-integrable on $[a, b]$, if there exists a continuous function F, where $F' = f$ almost everywhere in $[a, b]$, and $F \in CLC[a, b]$.

Based on the fact that if $F : [a, b] \to \mathbb{R}$ is a countably Lipschitz function, then $F \in ACG[a, b]$ [2], a CL-integrable function f on $[a, b]$ is Denjoy integrable in the wide sense on $[a, b]$ [2]. The Denjoy integral in the wide sense is more general than the Denjoy integral in the restricted sense. The Denjoy in the restricted sense on $[a, b]$ is equivalent to the Henstock-Kurzweil integral on $[a, b]$. It will be discussed sufficient conditions of the CL-integrable function to be Henstock-Kurzweil integrable function by using strongly bounded variation property. Furthermore, it will be given a strong version of CLC to have ACG^*.

The main tool in proving the theorems in Main Results is Lemma 1.5.

Lemma 1.5. *[5] Let X be a closed set in $[a, b]$ and $(a, b) \setminus X$ the union of (c_k, d_k) for $k = 1, 2, 3, \ldots$. If F is continuous on $[a, b]$, then the following conditions are equivalent:*

(i) $F \in AC^*(X)$,

(ii) $F \in AC(X)$ *and* $\sum_{k=1}^{\infty} \omega(F; [c_k, d_k]) < \infty$,

(iii) Definition of AC in Definition 1.2 holds with $a_i \in X$ or $b_i \in X$, for every i.

2. Main Results

If we consider Definition 1.1 and Definition 1.4, it can be seen that the primitive of a Henstock-Kurzweil integrable function is ACG^* and the primitive of a CL-integrable function is CLC. To have a CL-integrable function on $[a, b]$ to be Henstock-Kurzweil integrable on $[a, b]$, it is enough to prove the condition of countably Lipschitz to be ACG^* on $[a, b]$. The proof will be done by using Lemma 1.5 and the concept of strongly bounded variation as stated in Definition 2.2.

Lemma 2.1. *Let $X \subseteq [a, b]$ be a closed set and $(a, b) \setminus X = \cup(c_k, d_k)$. If a real-valued function F satisfies Lipschitz condition on $[a, b]$ and $\sum \omega(F; [c_k, d_k]) < \infty$, then $F \in AC^*(X)$.*

Proof. A Lipschitz function F on $[a, b]$ is an absolutely continuous on $[a, b]$. By Lemma 1.5, $F \in AC^*(X)$. ■

Definition 2.2. Let $X \subseteq [a, b]$. A real-valued function $F : [a, b] \to \mathbb{R}$ is said to be $VB^*(X)$ if

$$\sup \sum_{k} \omega(F; [a_k, b_k]) < \infty,$$

where ω denotes the oscillation of F on $[a_k, b_k]$ and the supremum is taken over all finite or infinite sequence of non-overlapping intervals $\{[a_k, b_k]\}$ with $a_k, b_k \in X$ for all k.

Theorem 2.3. *If a continuous real-valued function F has countably Lipschitz condition on $[a, b]$ and $F \in VB^*[a, b]$, then $F \in ACG^*[a, b]$.*

Proof. The function F is continuous and satisfies countably Lipschitz condition on $[a, b]$. There exists a collection of sets $\{E_n\}$, such that $[a, b] = \cup_n E_n$ and F satisfies Lipschitz condition on E_n, for every positive integer n.

That means, for every positive integer n, there is $M_n > 0$, such that for every $x, y \in E_n$, we have

$$|F(x) - F(y)| \leq M_n|x - y|.$$

Let $\epsilon > 0$ be given. Since F is continuous on $[a, b]$, then there exists a positive constant δ such that for any $x, y \in [a, b]$, if $|x - y| < \delta$, then $|F(x) - F(y)| < \epsilon$.

Fix n and since $F \in BV^*[a, b]$, then $F \in BV^*((a, b) \setminus E_n)$. By Lemma 1.5, it is enough to prove that $F \in AC(E_n)$. Put $\eta = \dfrac{\epsilon}{M_n}$. As corollary, for every finite or infinite collection of non-overlapping intervals $\{[c_k, d_k]\}$ with $c_k, d_k \in E_n$, if $\sum_k |d_k - c_k| < \eta$, we have

$$\begin{aligned}\sum_k \omega(F; [c_k, d_k]) &= \sum_k M_n|F(d_k) - F(c_k)| \\ &\leq M_n \sum_k |d_k - c_k| < \epsilon.\end{aligned}$$

■

Corollary 2.4. *If f is CL-integrable on $[a, b]$ with its primitive F has strongly bounded variation on $[a, b]$, then f is Henstock-Kurzweil integrable on $[a, b]$.*

The condition strongly bounded variation in Theorem 2.3 is still true if we replace it by VBG^*.

Definition 2.5. Let $X \subseteq [a, b]$. A real-valued function F is said to be $VBG^*(X)$ if there exists a collection of sets $\{X_n\}$ such that $F \in VB^*(X_n)$, for every n.

Theorem 2.6. *If a continuous real-valued function F has countably Lipschitz condition on $[a, b]$ and $F \in VBG^*[a, b]$, then $F \in ACG^*[a, b]$.*

Proof. Since F satisfies countably Lipschitz condition, there exists a collection $\{X_n\}$, such that $[a, b] = \cup_n X_n$ and F satisfies Lipschitz condition on X_n for every n. That means, for every n, there exists $M_n > 0$ such that for every $x, y \in X_n$,

$$|F(x) - F(y)| \leq M_n|x - y|.$$

The interval $[a, b] = \cap_m E_m$ with $F \in VB^*(E_m)$ for every m. Let us define $D_{nm} = X_n \cap E_m$, for every n and for every m. We have $[a, b] = \cup_m \cup_n D_{nm}$. The function F is Lipschitz on D_{nm}, so it is absolutely continuous on D_{nm}, for every n and m. By Lemma 1.5, $F \in AC^*(D_{nm})$, for every n and m. That means, $F \in ACG^*[a, b]$. ■

Corollary 2.7. *If f is CL-integrable on $[a, b]$ with its primitive F has strongly generalized bounded variation on $[a, b]$, then f is Henstock-Kurzweil integrable on $[a, b]$.*

The primitive of the CL-integral satisfies the countably Lipschitz condition and the primitive of the Henstock-Kurzweil integral is strongly generalized absolutely continuous. The modification of the countably Lipschitz condition to be countably Lipschitz condition in strong sense will give a sufficient condition of of a function to be strongly generalized absolutely continuous.

Definition 2.8. Let $X \subseteq \mathbb{R}$. A function $F : X \to \mathbb{R}$ is said to have strongly countably Lipschitz condition or countably Lipschitz condition in the strong sense on $[a, b]$, if there exists a countable collection of sets $\{X_n\}$, such that $X = \cup X_n$ and for every n, there exists $M_n > 0$, such that for every $x \in X_n$ or $y \in X_n$, we have

$$|F(x) - F(y)| \leq M_n|x - y|.$$

From the definition, it is clear that countably Lipschitz condition in the strong sense on $[a, b]$ implies countably Lipschitz condition on $[a, b]$. Every Lipschitz function is a countably Lipschitz in the strong sense. The function $F : \mathbb{R} \to \mathbb{R}$, where $F(x) = x^2$ satisfies countably Lipschitz condition in the strong sense on $\mathbb{R}$, but it does not satisfy Lipschitz condition on $\mathbb{R}$.

Lemma 2.9. *If the function $F : X \to \mathbb{R}$ satisfies countably Lipschitz condition in the strong sense on X, then F is continuous on X.*

Proof. Since the function F satisfies countably Lipschitz condition in strong sense on $[a, b]$, there exists a countable collection of sets $\{X_n\}$, such that $X = \cup X_n$ and for every n, there exists $M_n > 0$, such that for every $x \in X_n$ or $y \in X_n$, we have

$$|F(x) - F(y)| \leq M_n|x - y|.$$

Let x be an arbitrary point in X and $\epsilon > 0$. There is an n_0, with $x \in X_{n_0}$. Put $\delta = \frac{\epsilon}{M_{n_0}}$. As corollary, for every $y \in X$, if $|y - x| < \delta$, then

$$|F(y) - F(x)| \leq M_{n_0}|y - x| < M_{n_0}\delta = \epsilon.$$

■

Theorem 2.10. *If a real-valued function $F : [a, b] \to \mathbb{R}$ has countably Lipschitz condition in strong sense on $[a, b]$, then $F \in ACG^*[a, b]$.*

Proof. Since F has countably Lipschitz condition in strong sense on $[a, b]$, there exists a countable collection of sets $\{X_n\}$, with $[a, b] = \cup_n X_n$ and for every n, there exists an $M_n > 0$ such that for every x, y, where $x \in X_n$ or $y \in X_n$, we have

$$|F(x) - F(y)| \leq M_n|x - y|.$$

Let n be an arbitrary positive integer. Let $\epsilon > 0$ be given. Put $\delta = \dfrac{\epsilon}{M_n}$. As corollary, for every collection of non-overlapping interval $\{[a_{nk}, b_{nk}]\}$, where $x_{nk} \in X_n$ or $y_{nk} \in X_n$, if $\sum_k |x_{nk} - y_{nk}| < \delta$, then

$$\sum_k |F(x_{nk}) - F(y_{nk})| \leq \sum_k M_n|x_{nk} - y_{nk}| < M_n\delta = \epsilon.$$

By Lemma 2.9, F is continuous on $[a, b]$. Furthermore, by Lemma 1.5, F is absolutely continuous on X_n. As corollary, $F \in ACG^*[a, b]$. ■

3. Concluding Remarks

The sufficient conditions from CLC to be ACG^* have been done in Theorem 2.3 and Theorem 2.6. Those two theorems imply the countably Lipschitz function on $[a, b]$ is Henstock-Kurzweil integrable on $[a, b]$ as stated in Corollary 2.4 and Corollary 2.7, respectively. Moreover, the strongly countably Lipschitz condition on $[a, b]$ implies strongly generalized absolutely continuous on $[a, b]$ (Theorem 2.10).

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