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# Construction of a Banach Algebra in Summable Infinite Matrices and its Realization of the Gelfand Transformation

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**Abstract** In this paper, we extend the construction of a Banach Algebra from summable sequences to summable matrices. We also find the representation of the Banach algebra as complex-valued continuous functions defined on a Hausdorff space.

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## 1. Introduction

The original study of Banach algebra focused on the abstract structure of all bounded operators on Hilbert space [1]. In addition to the vector space with norms, we can define a multiplication operation in that set which is presented as a composition of the operator. The norm satisfies

$$\|AB\| \leq \|A\| \; \|B\|$$

where $A, B$ are two operators in the set.

The next question is whether the algebraic structure with that norm only occurs in the set of bounded operators. In the book [2], it is shown that this structure occurs in the space of summable sequences, i.e., the set

$$l^1(\mathbb{Z}) = \left\{ (x_i)_{i=-\infty}^{\infty} \,\middle|\, \sum_{i=-\infty}^{\infty} |x_i| < \infty, x_i \in \mathbb{C}, i \in \mathbb{Z} \right\}$$

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where $\mathbb{C}$ is the set of complex numbers and $\mathbb{Z}$ is the set of whole numbers. The space forms a Banach Algebra with a multiplication operation in the form of convolution and the norm used is

$$\|x\| = \sum_{i=-\infty}^{\infty} |x_i|\,,$$

where the meaning of the summation is the sum of two infinite series.

In this paper, we will construct a Banach algebra in the set of matrices of infinite order with two indices, where the sum of the absolute values of all terms is finite. Furthermore, based on Gelfand's theorem, all Banach algebras can be represented as the set of complex-valued continuous functions defined on some Hausdorff space. Here, we will identify the set of complex-valued continuous functions and the Hausdorff space.

## 2. Main Results

We begin by defining the set of our object as follows:

$$l^1(\mathbb{Z}\times\mathbb{Z}) = \left\{ (x_{ij})_{i,j=-\infty}^{\infty} \;\middle|\; \sum_{i=-\infty}^{\infty}\sum_{j=-\infty}^{\infty} |x_{ij}| < \infty, x_{ij}\in\mathbb{C}, i,j\in\mathbb{Z} \right\}$$

In the following discussion, we will refer to elements of this set as $x = (x_{ij})$ without explicitly stating the value of the indices.

Next, we will define the operations of summation and scalar multiplication as follows:

**Proposition 2.1.**

*(1) Let $x = (x_{ij})$, $y = (y_{ij})$ be the two element in $l^1(\mathbb{N}\times\mathbb{N})$, then $x + y = (x_{ij} + y_{ij})$ is also in $l^1(\mathbb{Z}\times\mathbb{Z})$.*
*(2) Let $x = (x_{ij})$ be an element in $l^1(\mathbb{Z}\times\mathbb{Z})$, and be a complex number $\alpha\in\mathbb{C}$, then $\alpha x = (\alpha x_{ij})$ is also in $l^1(\mathbb{Z}\times\mathbb{Z})$.*

The proof of this proposition is straightforward and can be shown using similar steps as the proof for the operation on $l^1(\mathbb{Z})$. With these properties, the set has a vector space structure over the scalars $\mathbb{C}$.

Next, we will define the norm of an element in $l^1(\mathbb{Z}\times\mathbb{Z})$. For $x = (x_{ij})$, we define the norm as:

$$\|x\| = \sum_{i=-\infty}^{\infty}\sum_{j=-\infty}^{\infty} |x_{ij}|$$

Note that the indices of the double sum are now allowed to range over both positive and negative integers.

We can easily prove that the number satisfies the properties of norm

**Proposition 2.2.**

*(1) If $x\in l^1(\mathbb{Z}\times\mathbb{Z})$, then $\|x\|\geq 0$.*
*The equality $\|x\| = 0$ if only if $x_{ij} = 0$ for every $i,j\in\mathbb{Z}$.*
*(2) For every $\alpha\in\mathbb{C}$ and $x\in l^1(\mathbb{Z}\times\mathbb{Z})$, sastisfy $\|\alpha x\| = |\alpha|\,\|x\|$.*
*(3) For every $x,y\in l^1(\mathbb{Z}\times\mathbb{Z})$ satisfy $\|x+y\|\leq\|x\|+\|y\|$.*

Next, we will define multiplication operation between the elements in the set. For $x = (x_{ij})$ and $y = (y_{ij})$, we define the multiplication as

$$(x * y)(n, m) = \sum_{i=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} x(n-i, m-j)\, y(i, j)$$

For one variable, this operation is known as convolution.

The next theorem states that the convolution operation for matrix with two indexes is well defined in the set. In one variable the inequality is know as Young Inequality.

**Theorem 2.3.** *For evert $x, y \in l^1(\mathbb{Z} \times \mathbb{Z})$, then*

$$\|x * y\| \leq \|x\| \, \|y\|$$

*Proof.* First, we count the norm $x * y$, that is

$$\begin{aligned} \|x * y\| &= \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} |(x * y)(n, m)| \\ &= \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \left| \sum_{i=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} x(n-i, m-j)\, y(i, j) \right|. \end{aligned}$$

Then we change the order of summation to have

$$\begin{aligned} \|x * y\| &\leq \sum_{i=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} |y(i, j)| \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} |x(n-i, m-j)| \\ &= \sum_{i=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} |y(i, j)| \, \|x\| \\ &= \|x\| \sum_{i=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} |y(i, j)| = \|x\| \, \|y\| . \end{aligned}$$

We have shown the inequality in the theorem. ∎

The Banach algebra $l^1(\mathbb{Z} \times \mathbb{Z})$ have unital properties, that means there exists a unit element or multiplication identity element. By looking at the elements of $l^1(\mathbb{Z} \times \mathbb{Z})$ as complex valued function on $\mathbb{Z} \times \mathbb{Z}$, the unit element of the algebra is the characteristic function on $(0, 0)$.

Furthermore, the Gelfand transformation theorem states that a Banach Algebra can be presented as a subset of complex-valued continuous functions on some Hausdorf space. In this case we will find a compact Hausdorf space and the presentation of the element of Banach Algebra as a complex-valued continuous function defined on the compact Hausdorf space. To do that, let write $z$ and $w$ each as a characteristic function characteristic function on $\{(1, 0)\}$ and $\{(0, 1)\}$, respectively.

Furthermore, for $n \in \mathbb{N}$, by using multiplication one the set, we can give the meaning of $z^n$ and $w^n$, which are the characteristics function of $\{(n, 0)\}$ and $\{(0, n)\}$ respectively. Again, viewing the elements at $l^1(\mathbb{Z} \times \mathbb{Z})$ as a complex value function at $\mathbb{Z} \times \mathbb{Z}$, we can write $x \in l^1(\mathbb{Z} \times \mathbb{Z})$ as

$$x = \sum_{i,j} x(i,j)\, z^i w^j$$

namely as a complex function of two variables $z, w$. Using these results, we can make a representation of the Gelfand transformation from the element of the algebra into set of complex-valued continuous functions. Suppose $z, w \in \mathbb{T}$ are the elements in the unit circle $\mathbb{T}$, then we define the mapping from Banach algebra to complex numbers as

$$\tau(x; z, w) = \sum_n \sum_m x(n,m)\, z^n w^m \in \mathbb{C}.$$

Since $\sum_n \sum_m |x(n,m)| < \infty$, then $\tau_{z,w}(x)$ is well defined and

$$|\tau(x; z, w)| \le \|x\|.$$

Next, it will be shown that the function $\tau(x; z, w)$ is continuous in $(z, w)$. But this is easy, because $\mathbb{T} \times \mathbb{T}$ is compact, and

$$\sum_{|n| \le N} \sum_{|m| < M} x(n,m)\, z^n w^m$$

is continuous for every $N, M \in \mathbb{N}$, then the function converges uniformly to $\tau(x; z, w)$.

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