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Subordinateness of the Subclass of Bazilevič Functions $B_2(\alpha)$

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Abstract In 1999 Marjono published about the subordinateness of the subclass of Bazilevic functions. However, the value of $\beta(\gamma)$ is not the largest one. In this paper, we present our work with my student Estelita Maria from Universidade Timor Loro Sa'e and Rahmalia Firdausi Tamara to find the largest $\beta(\gamma)$. This result was obtained by using a similar way to the theorem before and stressing an optimized approach and also by using the maximum modulus principles as mentioned in Remark 1.9. For an analytic, normalized function f such that $f(0) = f'(0) - 1 = 0$, there is a largest number $\beta^*(\gamma)$ such that subordination holds for $\alpha > 0$, $0 < \gamma \leq 1$ and $z \in D$ as an improvement of Marjono's work.

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1. Introduction

We first recall the Bazilevic functions $B(\alpha)$. In 1955, Bazilevič [2] introduced the so-called Bazilevič univalent functions using a differential equation of Löewner and Kufaref as follows:

Definition 1.1. [2]. Let f be analytic in the unit disc $D = \{z : |z| < 1\}$, and normalized such that $f(0) = f'(0) - 1 = 0$. Then f is called a Bazilevič function if there exists $g \in St$ such that for each $\alpha > 0$ and δ real,

$$Re\left[f'(z)\left(\frac{f(z)}{z}\right)^{\alpha+i\delta-1}\left(\frac{g(z)}{z}\right)^{-\alpha}\right] > 0, \qquad (1.1)$$

whenever $z \in D$, and St is the class of normalized starlike univalent functions.

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The case $\delta = 0$ is normally denoted by $B(\alpha)$ and has received considerable attention during the last forty-five or so years. See e.g. Thomas [14] and Sheil-Small [12]. We note that when $\alpha = 1$ we obtain $B(1) = K$.

Marjono [6], Thomas, Tuneski and Vasudevarao [16] and Asih [1] considered the subclass $B_1(\alpha)$ consisting of those functions in $B(\alpha)$ with $g(z) \equiv z$. This class was first considered by Singh [13] and has been the subject of several papers, e.g. Thomas [15], Obradović and Owa [8]. It follows easily from (1.1) that $f \in B_1(\alpha)$ if, and only if, for $z \in D$,

$$Re\left[f'(z)\left(\frac{f(z)}{z}\right)^{\alpha-1}\right] > 0.$$

Marjono [6] proved the following theorem for the function $f \in B_1(\alpha)$:

Theorem 1.2. *Let f be analytic in the unit disc $D = \{z : |z| < 1\}$, and f normalized such that $f(0) = f'(0) - 1 = 0$. Then for $\alpha > 0$, $0 < \gamma \le 1$, and $z \in D$, there exists $\beta(\gamma)$ such that*

$$f'(z)\left(\frac{f(z)}{z}\right)^{\alpha-1} \prec \left(\frac{1+z}{1-z}\right)^{\beta(\gamma)}$$

then

$$\left(\frac{f(z)}{z}\right)^{\alpha} \prec \left(\frac{1+z}{1-z}\right)^{\gamma}, \tag{1.2}$$

in particular we may choose

$$\beta(\gamma) = \gamma + \frac{2}{\pi}\arctan\left(\frac{\gamma}{\alpha}\right). \tag{1.3}$$

$\beta(\gamma)$ given by (1.3) is the largest number such that (1.2) holds.

In this paper we are concerned with subordination on a subclass of Bazilevič functions $B_2(\alpha)$ introduced by Marjono [6]. The subclass $B_2(\alpha)$ consist of those functions in $B(\alpha)$ with $g(z) \equiv z + z^2/2$. It follows easily from (1.1) that $f \in B_2(\alpha)$ if, and only if, for $z \in D$,

$$Re\left[\frac{2f'(z)}{2+z}\left(\frac{f(z)}{z+z^2/2}\right)^{\alpha-1}\right] > 0. \tag{1.4}$$

Next, Marjono [6] obtained the following theorem for the function $f \in B_2(\alpha)$:

Theorem 1.3. *Let f be analytic in the unit disc $D = \{z : |z| < 1\}$, and f normalized such that $f(0) = f'(0) - 1 = 0$. Then for $\alpha > 0$, $0 < \gamma \le 1$, and $z \in D$, there exists $\beta(\gamma)$ such that*

$$\frac{2f'(z)}{2+z}\left(\frac{f(z)}{z+z^2/2}\right)^{\alpha-1} \prec \left(\frac{1+z}{1-z}\right)^{\beta(\gamma)}$$

implies

$$\left(\frac{f(z)}{z+z^2/2}\right)^{\alpha} \prec \left(\frac{1+z}{1-z}\right)^{\gamma}, \tag{1.5}$$

in particular we may choose

$$\beta(\gamma) = \gamma + \frac{2}{\pi}\arctan\left(\frac{3\gamma}{4\alpha} - \frac{\alpha}{3\gamma}\right). \tag{1.6}$$

$\beta(\gamma)$ given by (1.6) is not the largest number such that (1.5) holds.

Related to subordination, Littlewood [5] and Rogosinski [10] introduced the term and discovered the basic relations of subordination. Over the year a substantial theory has been developed, and subordination now plays an important role in Complex analysis, see e.g. Thomas, Tuneski, and Vasudevarao [16]. We will need the following definition of subordination denoted by $f \prec g$:

Definition 1.4. [3]. Let f and g be analytic in $D = \{z : |z| < 1\}$. Then f is said to be univalent subordinate to g, if g is univalent in D, $f(0) = g(0)$, and $f(D) \subset g(D)$.

Theorem 1.5 (Subordination principles)**.** [11]. *Suppose that f be analytic in the unit disc $D = \{z : |z| < 1\}$, g analytic and univalent in D, $f(0) = g(0)$, and $f(D) \subset g(D)$. Then theese properties are follow :*

$$\begin{aligned}
&(i) \quad f = g \circ \omega,\ \omega \text{ is analytic},\ |\omega(z)| \leq |z|,\\
&(ii) \quad |f'(0)| \leq |g'(0)|,\ \text{with equality for } \omega(z) = e^{i\alpha}z,\\
&(iii) \quad f(|z| < r) \subset g(|z| < r),\ \text{for all } r,\ 0 < r < 1.
\end{aligned}$$

We did not prove Theorem 1.5, but we may use the Schwarz Lemma when proving this theorem.

Lemma 1.6 (Schwarz Lemma)**.** [9]. *Suppose that a function f is analytic in D, and that it obeys condition $f(0) = 0$ and $|f(z)| \leq 1$ for every $z \in D$. Then $|f'(0)| \leq 1$ and $|f(z)| \leq |z|$ for every $z \in D$. Furthermore, unless f happens to be a function of the type $f(z) = cz$ in D, where c is a complex constant of modulus one, it is actually true that $|f'(0)| \leq 1$ and that $|f(z)| \leq |z|$ when $0 < |z| < 1$.*

We will also need the idea of harmonic functions as the following:

Theorem 1.7. [3]. *Let a function u be harmonic in a domain D. Suppose that there exists a point z_0 of D with the property that $u(z) \leq u(z_0)$ for every z in D. Then u is constant in D.*

Corollary 1.8. [3]. *Let D be a bounded in the complex plane, and let $u : \overline{D} \longrightarrow R$ be a continuous function that is harmonic in D. Then $u(z)$ attains its maximum at some point on the boundary.*

From Theorem 1.7 and Corollary 1.8, we obtain the following remark:

Remark 1.9. A function harmonic in D and continuous in $\overline{D}$ must attain its maximum and minimum values on the boundary.

2. Main Results

In this session, we will find the largest value for $\beta(\gamma)$ by maximizing $\beta(\gamma)$ for the whole related θ with Mapple, then determining the value of $\beta(\gamma)$ by using the similar way in proving Theorem 2.1.

We obtained the following theorem for the function $f \in B_2(\alpha)$:

Theorem 2.1. *Let f be analytic in the unit disc $D = \{z : |z| < 1\}$, and f normalized such that $f(0) = f'(0) - 1 = 0$. Then for $\alpha > 0$, $0 < \gamma \leq 1$, and $z \in D$, there exists $\beta(\gamma)$ such that*

$$\frac{2f'(z)}{2+z}\left(\frac{f(z)}{z+z^2/2}\right)^{\alpha-1} \prec \left(\frac{1+z}{1-z}\right)^{\beta(\gamma)}$$

implies

$$\left(\frac{f(z)}{z+z^2/2}\right)^{\alpha} \prec \left(\frac{1+z}{1-z}\right)^{\gamma}, \tag{2.1}$$

in particular we may choose

$$\beta(\gamma) = (\gamma+1)\pi/2. \tag{2.2}$$

$\beta(\gamma)$ given by (2.2) is the largest number such that (2.1) holds.

In proving Theorem 2.1, we used the Miller-Mocanu Lemma, which is a boundary version of the Clunie-Jack Lemma [4].

Lemma 2.2 (Miller-Mocanu Lemma)**.** [7]. Let p be analytic in D and q be analytic and univalent in $\bar{D}$, with $p(0) = q(0)$. If $p \not\prec q$, then there is a point $z_0 \in D$ and $\zeta_0 \in \partial D$ such that $p(|z| < |z_0|) \subset q(D)$, $p(z_0) = q(\zeta_0)$, and

$$z_0 p'(z_0) = k\zeta_0 q'(\zeta_0), \qquad \text{for} \quad k \geq 1.$$

We are now proving Theorem 2.1.

Proof. By writing $p(z) = \left(\frac{f(z)}{z+z^2/2}\right)^{\alpha}$, and then differentiating to z, we only need to show that $\frac{2(1+z)p(z)}{2+z} + \frac{zp'(z)}{\alpha} \prec \left(\frac{1+z}{1-z}\right)^{\beta}$ implies $p(z) \prec \left(\frac{1+z}{1-z}\right)^{\gamma}$, whenever $\beta = \beta(\gamma)$ is given by (1.6).

For $z \in D$, let $h(z) = [(1+z)/(1-z)]^{\beta}$ and $q(z) = [(1+z)/(1-z)]^{\gamma}$, so that

$$|\arg h(z)| < \frac{\beta\pi}{2} \qquad \text{and} \qquad |\arg q(z)| < \frac{\gamma\pi}{2}.$$

In proving $p \prec q$, it is enough to use the proof of Marjono [6], then it is necessary to prove that $\beta(\gamma)$ given by (2.2) is the largest one.
Let $p(z) = \left(\frac{1+z}{1-z}\right)^{\gamma}$, then from the minimum modulus principle for harmonic functions, it follows that

$$\inf_{|z|<1} \arg\left(\frac{2(1+z)p(z)}{2+z} + \frac{zp'(z)}{\alpha}\right)$$

is attained at some point $z = e^{i\theta}$, for $0 < \theta < 2\pi$. Thus with $z = e^{i\theta}$

$$\begin{aligned}\frac{2(1+z)p(z)}{2+z} + \frac{zp'(z)}{\alpha} &= \frac{2(1+z)}{2+z}\left(\frac{1+z}{1-z}\right)^{\gamma} + \frac{2\gamma z}{\alpha(1-z)^2}\left(\frac{1+z}{1-z}\right)^{\gamma-1} \\ &= \frac{2[3(1+\cos\theta)+i\sin\theta]}{5+4\cos\theta}\left(\frac{\sin\theta}{1-\cos\theta}\right)^{\gamma}\exp\left(\frac{i\gamma\pi}{2}\right) \\ &\quad - \frac{\gamma}{\alpha(1-\cos\theta)}\left(\frac{\sin\theta}{1-\cos\theta}\right)^{\gamma-1}\exp\left(\frac{i(\gamma-1)\pi}{2}\right).\end{aligned} \tag{2.3}$$

Simplification on (2.3), then its RHS becomes

$$\left(\frac{\sin\theta}{1-\cos\theta}\right)^{\gamma-1}\left(\frac{1}{1-\cos\theta}\right)\left[\frac{2\sin\theta[3(1+\cos\theta)+i\sin\theta]}{5+4\cos\theta}\exp\left(\frac{i\gamma\pi}{2}\right)\right.$$
$$\left.-\frac{\gamma}{\alpha}\exp\left(\frac{i(\gamma-1)\pi}{2}\right)\right],$$

i.e.

$$\left(\frac{2(1+z)p(z)}{2+z}+\frac{zp'(z)}{\alpha}\right)\Big/6\left(\frac{\sin\theta}{1-\cos\theta}\right)^{\gamma}\left(\frac{1+\cos\theta}{5+4\cos\theta}\right)$$

$$\begin{aligned}
&= \left[\left(1+\frac{i\sin\theta}{3(1+\cos\theta)}\right)\exp\left(\frac{i\gamma\pi}{2}\right)-\frac{\gamma(5+4\cos\theta)}{6\alpha\sin\theta(1+\cos\theta)}\exp\left(\frac{i(\gamma-1)\pi}{2}\right)\right]\\
&= \left[\left(1+\frac{i\sin\theta}{3(1+\cos\theta)}\right)\left(\cos\frac{\gamma\pi}{2}+i\sin\frac{\gamma\pi}{2}\right)\right.\\
&\qquad\left.-\frac{\gamma(5+4\cos\theta)}{6\alpha\sin\theta(1+\cos\theta)}\left(\cos\frac{(\gamma-1)\pi}{2}+i\sin\frac{(\gamma-1)\pi}{2}\right)\right]. \qquad (2.4)
\end{aligned}$$

Since

$$\cos\frac{(\gamma-1)\pi}{2}=\sin\frac{\gamma\pi}{2} \qquad \text{and} \qquad \sin\frac{(\gamma-1)\pi}{2}=-\cos\frac{\gamma\pi}{2},$$

we can write the RHS of (2.4) as follows:

$$=\left(1+\frac{i\sin\theta}{3(1+\cos\theta)}\right)\left(\cos\frac{\gamma\pi}{2}+i\sin\frac{\gamma\pi}{2}\right)-\frac{\gamma(5+4\cos\theta)}{6\alpha\sin\theta(1+\cos\theta)}\left(\sin\frac{\gamma\pi}{2}-i\cos\frac{\gamma\pi}{2}\right),$$

which is

$$\begin{aligned}
=&\ \cos\frac{\gamma\pi}{2}-\frac{\sin\theta}{3(1+\cos\theta)}\sin\frac{\gamma\pi}{2}-\frac{\gamma(5+4\cos\theta)}{6\alpha\sin\theta(1+\cos\theta)}\sin\frac{\gamma\pi}{2}\\
&+i\left\{\frac{\sin\theta}{3(1+\cos\theta)}\cos\frac{\gamma\pi}{2}+\sin\frac{\gamma\pi}{2}+\frac{\gamma(5+4\cos\theta)}{6\alpha\sin\theta(1+\cos\theta)}\cos\frac{\gamma\pi}{2}\right\},
\end{aligned}$$

i.e.

$$\begin{aligned}
=&\ \cos\frac{\gamma\pi}{2}\left[1-\tan\frac{\gamma\pi}{2}\left(\frac{\sin\theta}{3(1+\cos\theta)}+\frac{\gamma(5+4\cos\theta)}{6\alpha\sin\theta(1+\cos\theta)}\right)\right.\\
&\left.+i\left\{\tan\frac{\gamma\pi}{2}+\frac{\sin\theta}{3(1+\cos\theta)}+\frac{\gamma(5+4\cos\theta)}{6\alpha\sin\theta(1+\cos\theta)}\right\}\right].
\end{aligned}$$

Taking arguments, we have

$$\beta^*(\gamma,\theta)=\arg\left(\frac{2(1+z)p(z)}{2+z}+\frac{zp'(z)}{\alpha}\right)=\arctan\left(\frac{U^*}{V^*}\right), \qquad (2.5)$$

where

$$\begin{aligned}
U^* &= \tan\frac{\gamma\pi}{2}+\frac{\sin\theta}{3(1+\cos\theta)}+\frac{\gamma(5+4\cos\theta)}{6\alpha\sin\theta(1+\cos\theta)},\\
V^* &= 1-\tan\frac{\gamma\pi}{2}\left(\frac{\sin\theta}{3(1+\cos\theta)}+\frac{\gamma(5+4\cos\theta)}{6\alpha\sin\theta(1+\cos\theta)}\right).
\end{aligned}$$

From (2.5) and using a trigonometry function property, we obtain

$$\begin{aligned}\beta^*(\gamma,\theta) &= \frac{\gamma\pi}{2} + \arctan\left(\frac{\sin\theta}{3(1+\cos\theta)} + \frac{k\gamma(5+4\cos\theta)}{6\alpha\sin\theta(1+\cos\theta)}\right), \quad \text{for } k \geq 1,\\ &\geq \frac{\gamma\pi}{2} + \arctan\left(\frac{\sin\theta}{3(1+\cos\theta)} + \frac{\gamma(5+4\cos\theta)}{6\alpha\sin\theta(1+\cos\theta)}\right)\\ &= \frac{\gamma\pi}{2} + \arctan(W^*),\end{aligned}$$

where

$$W^* = \frac{\sin\theta}{3(1+\cos\theta)} + \frac{\gamma(5+4\cos\theta)}{6\alpha\sin\theta(1+\cos\theta)}. \tag{2.6}$$

By calculating the whole value of θ, we obtain that the maximum of W^* will occurs when $\theta = k\pi$ as described at Figure 1 below. Thus the maximum of $\arctan(W^*) = \pi/2$ and the maximum of $\beta^*(\gamma,\theta) = (\gamma+1)\pi/2$

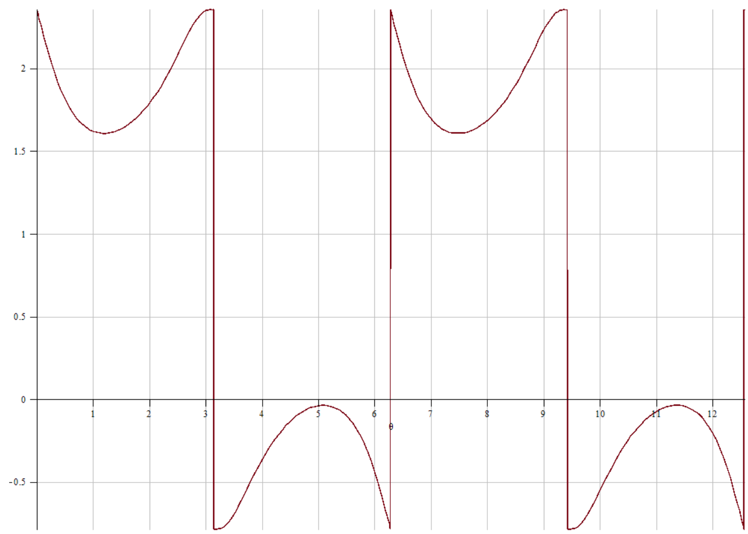

FIGURE 1. The value of $\Phi(\gamma,\theta)$ towards θ attains its maximum at $\theta = k\pi$

and this completes the proof. ∎

3. Concluding Remarks

We can improve the value of $\beta(\gamma)$ from $\beta(\gamma) = \gamma + \frac{2}{\pi}\arctan\left(\frac{3\gamma}{4\alpha} - \frac{\alpha}{3\gamma}\right)$ to the largest one $\beta(\gamma) = (\gamma+1)\pi/2$.

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