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On Invariant Eigenvalues of Complex Star Metric Graphs

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Abstract In this article, the eigenvalues of Laplacian acting on complex star metric graphs is considered. The operator is coupled with the Neumann-Kirchhoff vertex condition, implying the self adjointness of the operator. We exhibit the invariance of the eigenvalues over the number of the bonds of the star metric graphs. Moreover, the eigenvalues are also invariant over parallel bonds of the star metric multigraphs.

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1. Introduction

The advent of complex systems gives rise to the study of dynamics on large networks, where the hybrid (discrete-continuum) approach to discrete networks is more pronounced. One of the approaches is using Metric Graphs, or Quantum Graphs, which have become more mainstream in the last one or two decades. These studies appear naturally as a simplified model of physics and engineering when one considers dynamics or flow of various nature through quasi-one-dimensional objects, i.e. dynamics occurring in systems that look like a thin neighborhood of a graph. Modern nanoscience is full of those objects such as quantum wires, photonic crystals, carbon nano-structures, and thin waveguides. See for example Berkolaiko & Kuchment [1], Carini, Londergan, and Murdock [2]. Mathematically speaking, the study of such object involves 1D complexes, and differential operators,

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called Hamiltonian, acting on the complexes. Various identification, depending on the nature of the graphs, implies quite complicated matching conditions on vertices. Loosely speaking, the study of this nature is actually an inquiry into differential equations on each edge of the graph, satisfying various matching conditions at vertices.

One of the key studies of such objects is their spectral properties which are physically related to the energy structure of the systems. We study in particular the spectra and their properties of complex star metric graphs. Star graphs are important since they sit as a subgraph on any graph. Therefore knowing the spectral properties of a star is important to understand the global properties of the whole graph. Here we allow copiously bonds of a star graph, hence the term complex star graphs or star multigraphs.

Spectral studies of star metric graphs have been carried out in [3, 4] Spectral studies on general metric graphs are done in [1, 5]. For interactions between spectral properties and vertex condition, one may consult [4] For self-adjointness in metric graph one may find [1]. Finally for a more comprehensive review on metric graphs, or quantum graphs see for example [1, 5]. Computation of the spectra in the existing literature is usually done using scattering matrices of the metric graphs, coupled with some decomposition of operators, see [1, 6]. In this article, we compute the eigenvalues using a simpler and more accessible method, and hence more instructive for one starting to learn spectral studies of operators on metric graphs.

This article is organized as follows. We include some basic digression on graphs, metric graphs, and Hamiltonian operators, in Section 2. Some computation of spectra for star metric graphs is given in Section 3, along with invariance under change of directions, and addition of tentacles. Similar to what we do in Section 3, is repeated for Complex Star Graphs. In the end, we conclude our results with some remarks.

2. Laplace operator (or Laplacian) on metric graphs

A graph Σ is a pair $(\mathcal{V}, \mathcal{E})$, where $\mathcal{V} = \{v_i\}$ is the set of vertices, which can be finite or infinite; and $\mathcal{E} = \{e_j\}$ is the set of edges connecting the vertices. The notation $E = |\mathcal{E}|$ and $V = |\mathcal{V}|$ will be used to denote the number of edges, and the number of vertices, respectively. A graph is directed if, for each of its edges, a direction is assigned to it. In this case, each edge has one origin vertex and one terminal vertex. Such a graph is called a directional graph, or simply a digraph. Directed edges are called bonds. The set of all bonds is denoted by $\mathcal{B}$. The origin and the terminal vertices of a bond are specified by functions $o, t : \mathcal{B} \to \mathcal{V}$. In this case we say that bond b begins at $o(b)$ and ends at $t(b)$. The set of incoming bonds at a vertex v is the set of all bonds b such that $t(b) = v$. Similarly, we define the set of outgoing bonds at v. The orders of the set of incoming bonds, the set of outgoing bonds at v are denoted by d_v^i, d_v^o. Certainly, we have $d_v^i + d_v^o = d_v$, the degree of the vertex v.

Often a nondirectional graph G is considered as a digraph by assigning two bonds, b and $\bar{b}$ with opposite directions to each edge. The resulting digraph is denoted by $\tilde{G}$. The digraph $\tilde{G}$ is symmetric in the sense that $b \in \mathcal{B}$ if and only if there exists $\bar{b} \in \mathcal{B}$ such that $o(b) = t(\bar{b})$ and $o(\bar{b}) = t(b)$. The bond $\bar{b}$ is called the *reversal* of b.

A graph G is considered from two distinct perspectives. The first one is a standard object in graph theory and combinatorics, where an edge represents a relation between two vertices connected by the edge, rather than a physical link. In our context, G is considered a one-dimensional complex, where an edge is treated as a one-dimensional segment or an interval. In this point of view, an edge can be thought of as a physical wire, therefore the graph is now equipped with an additional metric structure.

A *metric graph* G is a directed graph, in which to each of bond b, an interval $[0, L_b]$ of length L_b is assigned, such that $o(b), t(b)$ correspond to $0, L_b$ respectively. A coordinate $x_b \in I_b = [0, L_b]$ is also assigned. This assignment endowed G with a metric structure, as the distance between a pair of vertices can be defined to be the minimum length of the path connecting them. With the coordinate function of each bond, the distance function can be now extended to any point x, y of the graph, which are not necessarily vertices.

Let a complex-valued function f be defined on the metric graph G, and let $f_b = f|_b$ be the restriction of f to the bond b. Well-definedness of f on G forces the *compatibility condition*, that is using slightly abused notation, for all the bond b incidence to vertex v, the value of all the restriction $f_b(v)$ at the vertex are identical. For a finite metric graph G, the Hilbert space $L^2(G)$ is defined to be the set of measurable functions f on G such that

$$\|f\|^2_{L^2(G)} := \sum_{b\in\mathcal{B}} \|f\|^2_{L^2[0,L_b]} < \infty,$$

so that $f \in L^2(G)$ if and only if $f_b \in L^2(I_b)$, for every bond b of G. The Sobolev space $H^1(G)$ is defined to be the space of continuous functions f on G, such that $f_b \in H^1(I_b)$, for each bond b, and such that

$$\|f\|^2_{H^1(G)} := \sum_{b\in\mathcal{B}} \|f_b\|^2_{H^1(I_b)}.$$

The Sobolev space $H^k(G)$ of higher order ($k > 1$), is harder to define, due to a lack of natural condition at vertices. Instead, we define

$$\tilde{H}^k(G) = \bigoplus_{b\in\mathcal{B}} H^k(I_b),$$

that is the space of function f for which the restriction on each bond b belongs to $H^k(I_b)$, regardless the condition defined at vertices.

We now define the Laplace operator (or Laplacian)

$$A : f \to -\frac{d^2 f}{dx^2},$$

on the space $\bigoplus_{b\in\mathcal{B}} \mathcal{C}_0^\infty$. We note that the operator has L^2-closure whose domain is $\tilde{H}^2_0(G)$. Furthermore, the operator is symmetric. Furthermore, if we consider the eigenvalue problem $Af = \lambda f$,

$$\langle Af, f\rangle = \langle -f'', f\rangle = \langle f', f'\rangle = \|f'\|^2_{L^2(G)} = \langle \lambda f, f\rangle = \lambda\|f\|^2_{L^2(G)}, \tag{2.1}$$

and thus nonnegativity of the eigenvalue is concluded. Here $\langle\cdot,\cdot\rangle$ is the L^2-inner product, in which integration by parts apply, and the boundary terms vanish due to the choice of the domain of the Laplacian.

3. Eigenvalue invariance in star metric graphs

We let the Laplace operator acts on the space $\bigoplus C^\infty(I_j)$. At each the vertex v, let the Neumann-Kirchhoff condition applies, that is $\sum_j f'_j(v) = 0$. The sum is taken in the set of indices j such that the bond I_j is incidence to the vertex v. This condition represents the conservation of current at junctions, hence the term Kirchhoff. However, at peripheral vertices (boundary of the graph), the sum reduces to a single term only, hence the Neumann boundary condition. Coupled with Neumann- Kirchhoff condition at vertices, Laplace operator is also symmetric.

We now consider a simple example of spectral property of a compact star graph. Let $\tilde{S}$ be a (metric) star graph with N bonds pointing towards the center C. With the hint of nonnegativity of eigenvalues, we solve

$$-\frac{d^2 f}{dx_j^2} = k^2 f(x_j),$$

with $f_j'(0_j) = 0$ for $j = 1, ..., N$, and $\sum_j f_j'(L_j) = 0$. It is now standard, to have $f_j(x_j) = A_j \cos(kx_j) + B_j \sin(kx_j), j = 1, \dots, N$. Upon substitution and imposing Neumann-Kirchhoff condition,

$$A_1 \cos(kL_1) = A_2 \cos(kL_2) = \cdots = A_N \cos(kL_N) = c \tag{3.1}$$

$$\sum_{j=1}^{N} A_j k \sin(kL_j) = 0 \tag{3.2}$$

Assuming $c \neq 0$, dividing the second equation by c, we conclude that k^2 is an eigenvalue if $\sum_{j=1}^{N} \tan(kL_j) = 0$. The case of $c = 0$ however, needs a separate consideration, see [1].

We contend that zero is an eigenvalue, since solving $-\frac{d^2 f}{dx_j^2} = 0$ on each bond, and using the Neumann condition at peripheral vertices, and compatibility conditions at the center yields

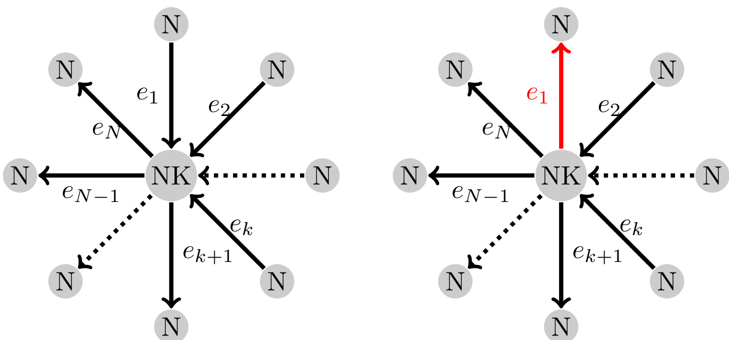
$$B_1 = B_2 = \cdots = B_N = c,$$

while Kirchhoff condition at the center is automatically satisfied. Therefore constant is an eigenfunction related to the zero eigenvalues. This argument holds for the Laplace operator with Neumann-Kirchhoff vertex condition in general metric graphs.

We establish the eigenvalue in variance property of the star metric graph over a change of direction of bonds.

Proposition 3.1. *The eigenvalue of the Laplacian on a star metric graph is invariant over a change of direction of its bonds.*

Proof. Let $\tilde{S}_N$ be a star metric graph with a given direction. Let $\tilde{S'}_N$ be $\tilde{S}_N$ with one of the bonds reversed, without loss of generality, let it be e_1.



By the positivity of the eigenvalue, we let $\lambda = \alpha^2$, for some $\alpha \in \mathbb{R}$. Then

$$f_1(x_1) = A_1 \cos(\alpha x_1) + B_1 \sin(\alpha x_1), \ x_1 \in I_1.$$

Reversing the direction yields change of parameter

$$x_1 \mapsto L - x_1,$$

and we let

$$F_1(x_1) = A_1' \cos(\alpha(L - x_1)) + B_1' \sin(\alpha(L - x_1)).$$

It is routine to check that

$$-F_1'' = \alpha F_1(x), f_1(0) = F_1(L), f_1(L) = F_1(0), f_1'(0) = -F_1'(L), f_1'(L) = -F_1'(0).$$

Therefore, starting from Neumann-Kirchhoff (NK) condition for $\tilde{S}'_N$,

$$\begin{aligned} -F_1'(0) + \sum_{j=2}^{k} f_j'(L) + \sum_{j=k+1}^{N} -f_j'(0) &= f_1'(L) + \sum_{j=2}^{k} f_j'(L) + \sum_{j=k+1}^{N} -f_j'(0) \\ &= \sum_{j=1}^{k} f_j'(L) + \sum_{j=k+1}^{N} -f_j'(0) = 0, \end{aligned}$$

which is the NK condition on $\tilde{S}_N$, hence λ is also an eigenvalue for $\tilde{S}'_N$. ∎

We utilize the following property of matrices to compute eigenvalues of $\tilde{S}_N$ later.

Theorem 3.2 (Zhang [7]).
Let M be a square $(m+n)\times(m+n)$ matrix, with decomposition $M = \begin{bmatrix} A_{m\times m} & B_{m\times n} \\ C_{n\times m} & D_{n\times n} \end{bmatrix}$. *If $D_{n\times n}$ is invertible,*

$$\det(M) = \det(A - BD^{-1}C)\det(D).$$

The fact that the eigenvalue of the Laplacian on the star metric graph is independent from the directions of the bonds, provides us some convenience in assuming a certain direction. We now establish the invariance of the eigenvalues of the Laplacian of star metric graph, over the number of bonds N.

Theorem 3.3. *The eigenvalues set of the Laplacian on star metric graph $\tilde{S}_N$, is independent of the number of the bonds N, for $N \geq 2$.*

Proof. By eigenvalue invariance we can choose a set of directions for the bonds, in this case, we choose the ones which are pointing toward the center. Apriorily, we let λ be an eigenvalue and $f = \langle f_1, \ldots, f_N \rangle$ be the corresponding eigenfunction. Again, by the positivity of eigenvalue, we let $\lambda = \alpha^2$. Then we have $-f_j'' = \lambda f_j$, for $j = 1, 2, ..., N$, for which the solution is $f_j(x_j) = A_j \cos(\alpha x_j) + B_j \sin(\alpha x_j)$. For the star metric graph $\tilde{S}_N$ the conditions on the vertices are the following.

$$f_1(L) = f_2(L) = ... = f_N(L) \tag{3.3}$$

$$f_1'(L) + f_2'(L) + ... + f_N'(L) = 0 \tag{3.4}$$

$$f_1'(0) = f_2'(0) = ... = f_N'(0) = 0. \tag{3.5}$$

The first one is the compatibility condition at the center, and the second one is the NK condition also at the center. The third one is the Neumann condition at the peripheral vertices. From the Neumann conditions at the periphery all $B_j = 0$, so now we have $f_j(x_j) = A_j \cos(\alpha x_j)$. Upon substituting $f_j, j = 1, \ldots, N$ into the above conditions yields coefficient matrix with $A_j, j = 1, \ldots, N$ as the unknowns. We contend that the determinant of the coefficient matrix for $\tilde{S}_N$ is $(-1)^{N-1} N \sin(\alpha L)\cos^{N-1}(\alpha L)$. We start with $N = 2$. The coefficient matrix for $\tilde{S}_2$, is:

$$M_2 = \begin{bmatrix} \sin(\alpha L) & \sin(\alpha L) \\ \cos(\alpha L) & -\cos(\alpha L) \end{bmatrix}$$

and the determinant is

$$\det(M) = -2\sin(\alpha L)\cos(\alpha L). \tag{3.6}$$

The star metric graph $\tilde{S}_N$ can be constructed from attachment of $N - 2$ intervals with Neumann conditions on both ends, to the center of $\tilde{S}_2$. Then the coefficient matrix for $\tilde{S}_N$ can be partitioned into

$$M_N = \begin{bmatrix} M_2 & E_c \\ F_c & D_c \end{bmatrix}$$

with $2 \times c$ matrix

$$E_c = \begin{bmatrix} \sin(\alpha L) & \sin(\alpha L) & \cdots & \sin(\alpha L) \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

a $c \times 2$ matrix

$$F_c = \begin{bmatrix} \cos(\alpha L) & 0 \\ \cos(\alpha L) & 0 \\ \vdots & \vdots \\ \cos(\alpha L) & 0 \end{bmatrix},$$

and a $c \times c$ matrix

$$D_c = -\cos(\alpha L) I_c.$$

The matrix D is invertible, with $D_c^{-1} = -\dfrac{1}{\cos(\alpha L)} I_c$, and hence

$$E_c D_c^{-1} F_k = -\frac{1}{\cos(\alpha L)} \begin{bmatrix} \alpha \sin(\alpha L)\cos(\alpha L) & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} -\alpha \sin(\alpha L) & 0 \\ 0 & 0 \end{bmatrix}.$$

Therefore

$$M_2 - E_c D_c^{-1} F_c = \begin{bmatrix} (1+c)\sin(\alpha L) & \sin(\alpha L) \\ \cos(\alpha L) & -\cos(\alpha L) \end{bmatrix},$$

and further $\det(M_2 - E_c D_c^{-1} F_c) = -(2+c)\sin(\alpha L)\cos(\alpha L)$. Using Proposition 3.1, $\det(M_N) = (-1)^{c+1}(2+c)\sin(\alpha L)\cos^{c+1}(\alpha L)$. Since $c = N - 2$, then $\det(M_N) = (-1)^{N-1} N \sin(\alpha L)\cos^{N-1}(\alpha L)$.

To obtain a nontrivial solution we set $\det(M_N) = 0$. We have $\alpha = \dfrac{(n - \frac{1}{2})\pi}{L}$, or $\alpha = \dfrac{n\pi}{L}$.

The eigenvalues are $\lambda = \dfrac{\left(n - \frac{1}{2}\right)^2 \pi^2}{L^2}$, and $\lambda = \dfrac{n^2\pi^2}{L^2}$, for $n \in \mathbb{Z}$.
The result is independent of the order of the equations used to derive the coefficient matrix since $|\det(M_N)|$ is unchanged. ∎

4. The Case of Star Metric Multigraphs

A multigraph $\tilde{MG}$ is a nonsimple graph such that there are edges having the same initial-terminal vertices, $\partial(e_i) = \partial(e_j)$. Geometrically speaking, in a multigraph, there are pairs of vertices connected by more than one edge. Those edges are called parallel. A star metric multigraph $\tilde{MS}_N$ is a multigraph such that if their parallel bonds are collapsed into one edge, then the star metric graph $\tilde{S}_N$ is obtained. A star metric multigraphs are denoted by $\widetilde{MS}_N\,(k_i)_{i=1}^N$, with $(k_i)_{i=1}^N$ is an N-tuple of positive integer, and k_i denotes the number of bonds parallel to e_i, for every $i = 1, 2, ..., N$. Then the star metric multigraph $\widetilde{MS}_N$ has $N + \sum_{i=1}^{N} k_i$ bonds. The Figure 1 below is an example of $\widetilde{MS}_3\,(1,1,2)$.

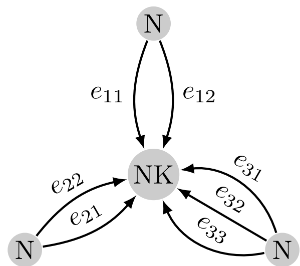


Figure 1. Metric Multigraph $\widetilde{MS}_3\,(1,1,2)$.

We will compute the eigenvalues of the Laplacian on $\widetilde{MS}_2\,(k_1, 0)$. The following theorem states that the eigenvalue of the Laplacian on $\widetilde{MS}_2\,(k_1, 0)$ is independent of the number of the parallel k_1.

Theorem 4.1.
The eigenvalue of the Laplacian on $\widetilde{MS}_2\,(k_1, 0)$ is invariant over the number of bonds parallel to I_1 at $MS_2\,(k_1, 0)$.

Theorem 4.1 is proved in a similar fashion as the previous one using the coefficient matrix obtained from vertex conditions of the metric graph applied to the function f; that is starting from star graph $\tilde{S}_2$ and adding repeatedly, on one parallel bond with a Neumann condition, at a time.

(1) Star graph $\tilde{S}_2$
First we will review $\tilde{S}_2$ in the following direction as shown in Figure 2:
Assuming $\lambda = \alpha^2$, the solution to the Laplacian eigenvalue problem on $\tilde{S}_2$ is

$$f_j(x_j) = A_j \cos(\alpha x_j) + B_j \sin(\alpha x_j)$$

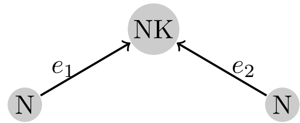


FIGURE 2. Metric star graph $\widetilde{S}_2$.

for $j = 1, 2$. Applying compatibility and NK conditions, a system of equations is obtained with A_1, A_2, B_1, B_2 as the unknowns. The related coefficient matrix is

$$M_2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\sin(\alpha L) & \cos(\alpha L) & -sin(\alpha L) & \cos(\alpha L) \\ \cos(\alpha L) & \sin(\alpha L) & -\cos(\alpha L) & -\sin(\alpha L) \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

so that,

$$\det(M_2) = -2\sin(\alpha L)\cos(\alpha L). \tag{4.1}$$

(2) Star metric multigraph $\widetilde{MS}_2\,(1,0)$

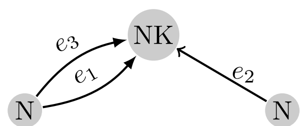


FIGURE 3. Star Multigraph $\widetilde{MS}_2\,(1,0)$).

The coefficient matrix is obtained as follows:

$$M_2(1) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 1 \\ -\sin(\alpha L) & \cos(\alpha L) & -sin(\alpha L) & \cos(\alpha L) & -\sin(\alpha L) & \cos(\alpha L) \\ \cos(\alpha L) & \sin(\alpha L) & -\cos(\alpha L) & -\sin(\alpha L) & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & -1 & 0 \\ \cos(\alpha L) & \sin(\alpha L) & 0 & 0 & -\cos(\alpha L) & -\sin(\alpha L) \end{bmatrix}.$$

Let

$$H = \begin{bmatrix} 0 & 1 \\ -\sin(\alpha L) & \cos(\alpha L) \end{bmatrix}, \quad D = \begin{bmatrix} -1 & 0 \\ -\cos(\alpha L) & -\sin(\alpha L) \end{bmatrix}$$

then $M_2(1)$ can be written as:

$$M_2(1) = \begin{bmatrix} M_2 & C_1 \\ E_1 & D_1 \end{bmatrix}$$

where M_2 is the previous coefficient matrix on the metric graph $\tilde{S}_2$, and

$$C_1 = \begin{bmatrix} H \\ 0_{2\times 2} \end{bmatrix}, \quad E_1 = \begin{bmatrix} -D & 0_{2\times 2} \end{bmatrix}, \quad D_1 = D.$$

Since the $\det(D_1) = \sin(\alpha L)$, so D_1 has an inverse. Consequently, according to the Theorem 3.2,

$\det(M_2(1)) = \det(M_2 - C_1 D_1^{-1} E_1)\det(D_1)$.

Next, we have

$$\begin{aligned} C_1 D_1^{-1} E_1 &= \begin{bmatrix} H \\ 0_{2\times 2} \end{bmatrix} D_1^{-1} \begin{bmatrix} -D_1 & 0_{2\times 2} \end{bmatrix} \\ &= \begin{bmatrix} H \\ 0_{2\times 2} \end{bmatrix} \begin{bmatrix} -I_{2\times 2} & 0_{2\times 2} \end{bmatrix} \\ &= \begin{bmatrix} -H & 0_{2\times 2} \\ 0_{2\times 2} & 0_{2\times 2} \end{bmatrix} \end{aligned}$$

so that,

$$\begin{aligned} M_2 - C_1 D_1^{-1} E_1 &= \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\sin(\alpha L) & \cos(\alpha L) & -\sin(\alpha L) & \cos(\alpha L) \\ \cos(\alpha L) & \sin(\alpha L) & -\cos(\alpha L) & -\sin(\alpha L) \\ 0 & 0 & 0 & 1 \end{bmatrix} \\ &- \begin{bmatrix} 0 & -1 & 0 & 0 \\ -\sin(\alpha L) & \cos(\alpha L) & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 2 & 0 & 0 \\ -2\sin(\alpha L) & 2\cos(\alpha L) & -sin(\alpha L) & \cos(\alpha L) \\ \cos(\alpha L) & \sin(\alpha L) & -\cos(\alpha L) & -\sin(\alpha L) \\ 0 & 0 & 0 & 1 \end{bmatrix}. \end{aligned}$$

Using the minor cofactor matrix, it is obtained

$\det(M_2 - C_1 D_1^{-1} E_1) = -6\sin(\alpha L)\cos(\alpha L)$

and as a result

$$\begin{aligned} \det(M_2(1)) &= \det(M_2 - C_1 D_1^{-1} E_1)\det(D_1) \\ &= -6\sin^2(\alpha L)\cos(\alpha L). \end{aligned} \tag{4.2}$$

(3) Star Multigraph $\widetilde{MS}_2\,(2,0)$

We proceed with $\widetilde{MS}_2\,(2,0)$ metric graph as the Figure 4. The coefficient matrix

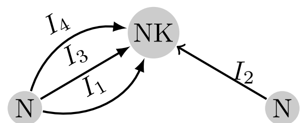


FIGURE 4. Star Multigraph $\widetilde{MS}_2\,(2,0)$).

is obtained, namely $M_2(2)$. It can be written as:

$$M_2(2) = \begin{bmatrix} M_2 & C_2 \\ E_2 & D_2 \end{bmatrix}$$

where M_2 is the coefficient matrix on the metric graph $\tilde{S}_2$ and,

$$C_2 = \begin{bmatrix} H & H \\ 0_{2\times 2} & 0_{2\times 2} \end{bmatrix}, E_2 = \begin{bmatrix} -D & 0_{2\times 2} \\ -D & 0_{2\times 2} \end{bmatrix},$$

$$D_2 = \begin{bmatrix} D & 0_{2\times 2} \\ 0_{2\times 2} & D \end{bmatrix}.$$

Note that $\det(D_2) = \det(D)\det(D) = \sin^2(kL)$ so D_2 has an inverse. Using Theorem 3.2

$$\det(M_2(2)) = \det(M_2 - C_2 D_2^{-1} E_2)\det(D_2).$$

Next

$$\begin{aligned} C_2 D_2^{-1} E_2 &= \begin{bmatrix} H & H \\ 0_{2\times 2} & 0_{2\times 2} \end{bmatrix} \begin{bmatrix} D^{-1} & 0_{2\times 2} \\ 0_{2\times 2} & D^{-1} \end{bmatrix} \begin{bmatrix} -D & 0_{2\times 2} \\ -D & 0_{2\times 2} \end{bmatrix} \\ &= \begin{bmatrix} H & H \\ 0_{2\times 2} & 0_{2\times 2} \end{bmatrix} \begin{bmatrix} -I_{2\times 2} & 0_{2\times 2} \\ -I_{2\times 2} & 0_{2\times 2} \end{bmatrix} \\ &= \begin{bmatrix} -2H & 0_{2\times 2} \\ 0_{2\times 2} & 0_{2\times 2} \end{bmatrix} \end{aligned}$$

so that,

$$\begin{aligned} M_2 - C_2 D_2^{-1} E_2 &= \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\sin(\alpha L) & \cos(\alpha L) & -sin(kL) & \cos(\alpha L) \\ \cos(\alpha L) & \sin(\alpha L) & -\cos(\alpha L) & -\sin(\alpha L) \\ 0 & 0 & 0 & 1 \end{bmatrix} \\ &- \begin{bmatrix} 0 & -2 & 0 & 0 \\ -2\sin(\alpha L) & 2\cos(\alpha L) & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 3 & 0 & 0 \\ -3\sin(\alpha L) & 3\cos(\alpha L) & -sin(kL) & \cos(\alpha L) \\ \cos(\alpha L) & \sin(\alpha L) & -\cos(\alpha L) & -\sin(\alpha L) \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{aligned}$$

and hence

$$\det(M_2 - C_2 D_2^{-1} E_2) = -12\sin(\alpha L)\cos(\alpha L).$$

Consequently,

$$\begin{aligned} \det(M_2(2)) &= \det(M_2 - C_2 D_2^{-1} E_2)\det(D_2) \\ &= -12\sin^3(\alpha L)\cos(\alpha L). \end{aligned} \tag{4.3}$$

From the process prescribed 1, 2, and 3, we conclude that if there are k_1 parallel intervals with I_1, then the coefficient matrix $M_2(k_1)$ can be written in the form:

$$M_2(k_1) = \begin{bmatrix} M_2 & C_{k_1} \\ E_{k_1} & D_{k_1} \end{bmatrix}$$

where C_{k_1} is a matrix of size $(4 \times 2k)$

$$C_{k_1} = \begin{bmatrix} H & H & \dots & H \\ 0_{2\times 2} & 0_{2\times 2} & \dots & 0_{2\times 2} \end{bmatrix}$$

and E_{k_1} is a matrix of size $(2k \times 4)$

$$E_{k_1} = \begin{bmatrix} -D & 0_{2\times 2} \\ -D & 0_{2\times 2} \\ \vdots & \vdots \\ -D & 0_{2\times 2} \end{bmatrix}$$

while D_{k_1} is a matrix of size $(2k \times 2k)$ which can be written in the form

$$D_{k_1} = \begin{bmatrix} D & 0_{2\times 2} & \dots & 0_{2\times 2} \\ 0_{2\times 2} & D & & 0_{2\times 2} \\ \vdots & & \ddots & \vdots \\ 0_{2\times 2} & 0_{2\times 2} & \dots & D \end{bmatrix}$$

with

$$H = \begin{bmatrix} 0 & 1 \\ -\sin(\alpha L) & \cos(\alpha L) \end{bmatrix}, \quad D = \begin{bmatrix} -1 & 0 \\ -\cos(\alpha L) & -\sin(\alpha L) \end{bmatrix}.$$

We observe that $\det(D_{k_1}) = \det(D)^k = \sin^k(kL)$ and then:

$$\begin{aligned} C_{k_1} D_{k_1}^{-1} E_{k_1} &= \begin{bmatrix} H & H & \dots & H \\ 0_{2\times 2} & 0_{2\times 2} & \dots & 0_{2\times 2} \end{bmatrix} \begin{bmatrix} D & 0_{2\times 2} & \dots & 0_{2\times 2} \\ 0_{2\times 2} & D & & 0_{2\times 2} \\ \vdots & & \ddots & \vdots \\ 0_{2\times 2} & 0_{2\times 2} & \dots & D \end{bmatrix} \begin{bmatrix} -D & 0_{2\times 2} \\ -D & 0_{2\times 2} \\ \vdots & \vdots \\ -D & 0_{2\times 2} \end{bmatrix} \\ &= \begin{bmatrix} H & H & \dots & H \\ 0_{2\times 2} & 0_{2\times 2} & \dots & 0_{2\times 2} \end{bmatrix} \begin{bmatrix} -I_{2\times 2} & 0_{2\times 2} \\ -I_{2\times 2} & 0_{2\times 2} \\ \vdots & \vdots \\ -I_{2\times 2} & 0_{2\times 2} \end{bmatrix} \\ &= \begin{bmatrix} -k_1 H & 0_{2\times 2} \\ 0_{2\times 2} & 0_{2\times 2} \end{bmatrix} \end{aligned}$$

so that

$$M_2 - C_{k_1} D_{k_1}^{-1} E_{k_1} = \begin{bmatrix} 0 & 1+k_1 & 0 & 0 \\ -(1+k_1)\sin(\alpha L) & (1+k_1)\cos(\alpha L) & -\sin(\alpha L) & \cos(\alpha L) \\ \cos(\alpha L) & \sin(\alpha L) & -\cos(\alpha L) & -\sin(\alpha L) \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Using the minor cofactor matrix,

$$\det(M_2 - C_{k_1} D_{k_1}^{-1} E_{k_1}) = -(1+k_1)(2+k_1)\sin(\alpha L)\cos(\alpha L)$$

so that

$$\det(M_{k_1}) = -(1+k_1)(2+k_1)\sin^{k_1+1}(\alpha L)\cos(\alpha L).$$

Now we compute λ. To obtain a non-trivial solution f, the condition $\det(M_{k_1}) = 0$ should be satisfied. As a result

$$\alpha = \frac{n\pi}{L} \text{ or } \alpha = \frac{\left(n - \frac{1}{2}\right)\pi}{L},$$

for $n \in \mathbb{Z}$. Since $\lambda = \alpha^2$, we get $\lambda = \dfrac{n^2\pi^2}{L^2}$ or $\lambda = \dfrac{\left(n - \frac{1}{2}\right)^2 \pi^2}{L^2}$. ■

In a similar way, it can be proved that the eigenvalues of the Laplacian on $\widetilde{MS}_2(0, k_2)$ is invariant over the number of intervals parallel to I_2.

By invariance and symmetry, the eigenvalues at $\widetilde{MS}_2(k_1, 0)$ and $\widetilde{MS}_2(0, k_1)$ in any direction are $\lambda = \dfrac{\left(n - \frac{1}{2}\right)^2 \pi^2}{L^2}$ and $\lambda = \dfrac{n^2\pi^2}{L^2}$, for $n \in \mathbb{Z}$.

Corollary 4.2. *Eigenvalues of $\widetilde{MS}_2(0, k_2)$ as well as $\widetilde{MS}_2(k_1, k_2)$ are*

$$\lambda = \frac{\left(n - \frac{1}{2}\right)^2 \pi^2}{L^2} \quad \text{and} \quad \lambda = \frac{n^2\pi^2}{L^2},$$

for $n \in \mathbb{Z}$.

5. Concluding Remarks

We have computed the eigenvalues of the Laplacian on star metric graphs, as well as star metric multigraphs. Invariant properties of the eigenvalues over the variations of directions, as well as over the number of bonds of the star graphs enable one to compute the eigenvalue using the simplest structure,i.e. $\tilde{S}_2$. However the method we employ so far, is not able to reveal the multiplicities of the eigenvalues since we derive the eigenvalues from the determinant of the coefficient matrix, that is an equation in trigonometric terms $\sin, \cos$, and not from the characteristic polynomial. Therefore the spectral studies carried out using this method are not complete yet. A further study of pushing this method to obtain information on how large the eigenspaces are is an interesting further exploration.

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