

Mathematical Study of One Prey and Two Competing Predators Considering Beddington-DeAngelis Functional Response with Distributed Delay

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Abstract A nonlinear prey-predator mathematical model is suggested and discussed. In the model, one prey and two competing predators, and Beddington-DeAngelis functional response with distributed delay have been considered. The analysis of the stability of models is derived and sufficient controls and suitable rules have been discussed for the system. All the feasible equilibria of the system are obtained and the conditions are determined. In the study, it has been observed that without distributed delay, the system would reach a stable position. Additionally, we have supported our analytical conclusions with numerical visualizations.

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1. Introduction

Ecology is the study of how a community of creatures interacts with its surroundings, including how it uses resources and handles rivals. To achieve the highest level of Darwinian fitness, every population should take advantage of a distinctive, dynamic, and diverse environment. Understanding interactions between predators and their prey is the major goal of ecology, and one important aspect of these interactions is how quickly an average predator consumes its food [38].

The description of the relation among species began when Lotka created a set of non-linear equations of the first order that illustrated the effect, particularly of a parasite on prey, also Vito Volterra took into account the same prey-predator relationship population [16]. In the year 1939, Leslie started working in the field of biomathematics, and he made a contribution to the modeling of predatory prey species. In addition to this, the



researchers Richards and Hopkins proposed an additional general mathematical model formulation in 1959, that led to Lotka Volterra mathematical predator-prey expressions. Along with Lotka and Volterra's mathematical model, another researcher Kolmogorov discussed Mendel's regulations and got through with some theories that established a system of differential equations to discuss the predator-prey mathematical model, particularly in a certain environment [2].

In the past, many researchers have studied widely the survival and the extinction of prey-predator populations. It is noticed that in some occasions chiefly when the predator populations hunt food mainly for survival and to share/fight, a very acceptable and practical prey-predator population theory must be available in the dependent ratio approach so that the predator population growth rate must be a function of prey population's ratio to that of predator abundance, also the professed as predator functional response [1, 3, 6, 7]. It was demonstrated that the prey-predator ratio dependent mathematical models have the potential to reproduce richer and plausible or tolerable dynamics [5, 12].

The predator-prey models with three species captivated many researchers [4, 13, 15, 17, 20, 22, 24, 27–29, 32]. The interactivity can be elucidated between more than two species in one single place. In some representations, we study one prey and two predator mathematical interactions models where two types of predators feed the same prey, due to the intrinsic nature of the predators, they struggle each time to seize the prey [4, 13, 15, 17, 18, 20–22, 24, 27–29, 32]. In very recent years, along with the prey-predator model many authors have considered the Beddington DeAngelis mathematical functional models to study the delays, help, and stochastic environment [33, 37–39].

The ecological environment and behavior realistically change when the time delays are incorporated in the population of prey-predator differential equations [8–11, 23]. This model is established on the supposition that in the physical world prey populations do not grow exponentially in the absence of predators, but their size will be restricted by the absence of resources in due course. The delay corresponds to the interval between the predator apprehending the prey and the reformation to biomass for predators. The growth of a population subject to maturation delay is modeled using either a discrete or a continuous delay. Many ecologists and mathematicians have enormously studied the prey-predator mathematical models by considering the distributed delay's [35, 36] researchers in their models [14].

In this paper, we have considered a nonlinear mathematical model of one prey and two competing predators and incorporated the function of Beddington-DeAngelis response, and studied the effect of distributed delay in the model. Numerical demonstrations are illustrated with the help of MATLAB, and the importance of distributed delay has been highlighted.

2. Mathematical model

Here N is the prey population and y_1 and y_2 are the predator populations. We assume a_1 and a_2 as predation rates of prey by y_1 and y_2 respectively. The conversion efficiencies are taken as b_1 and c_1 for y_1 and y_2 respectively. The natural death rates of y_1 and y_2 are taken as b_2 and c_2 respectively. The terms $\alpha_1 y_2$ and $\alpha_2 y_1$ in the denominator

determine the effects of the mutual interference between predators. Moreover, r is considered as growth rate and k is carrying capacity. Here $(N(t))/(1 + N(t) + \alpha_1 y_2)$ and $(N(t))/(1 + N(t) + \alpha_2 y_1)$ are so called function of Beddington DeAngelis response.

Considering variables as $N(t)$, $y_1(t)$, and $y_2(t)$, we formulate the following model:

Model-A:

$$\begin{aligned}\frac{dN}{dt} &= rN \left(1 - \frac{N}{k}\right) - \frac{\alpha_1 N y_1}{1 + N + \alpha_1 y_2} - \frac{\alpha_2 N y_2}{1 + N + \alpha_2 y_1} \\ \frac{dy_1}{dt} &= \frac{b_1 \alpha_1 N y_1}{1 + N + \alpha_1 y_2} - b_2 y_1 \\ \frac{dy_2}{dt} &= \frac{c_1 \alpha_2 N y_2}{1 + N + \alpha_2 y_1} - c_2 y_2\end{aligned}\tag{2.1}$$

with $N(0) \geq 0$, $y_1(0) \geq 0$, and $y_2(0) \geq 0$.

Introducing delay (distributed) in the metabolism of prey population by the predator populations [19, 30], then the equations are:

$$\frac{dy_1}{dt} = \frac{b_1 \alpha_1 N y_1}{1 + N + \alpha_1 y_2} \int_{-\infty}^t \beta_1 f(s) e^{(-\beta_1(t-s))} ds - b_2 y_1$$

where $f(s) = f(N) = N(s)$, β_1 is delay parameter,

$$\frac{dy_2}{dt} = \frac{c_1 \alpha_2 N y_2}{1 + N + \alpha_2 y_1} \int_{-\infty}^t \beta_2 f(u) e^{(-\beta_2(t-u))} du - c_2 y_2$$

where $f(u) = f(N) = N(u)$, β_2 is delay parameter, and substituting

$$\bar{R}_1(t) = \int_{-\infty}^t \beta_1 f(s) e^{(-\beta_1(t-s))} ds$$

and

$$\bar{R}_2(t) = \int_{-\infty}^t \beta_2 f(u) e^{(-\beta_2(t-u))} du$$

then Model-A (2.1) becomes:

Model-B:

$$\begin{aligned}\frac{dN}{dt} &= rN \left(1 - \frac{N}{k}\right) - \frac{\alpha_1 N y_1}{1 + N + \alpha_1 y_2} - \frac{\alpha_2 N y_2}{1 + N + \alpha_2 y_1} \\ \frac{dy_1}{dt} &= \frac{b_1 \alpha_1 \bar{R}_1 y_1}{1 + N + \alpha_1 y_2} - b_2 y_1 \\ \frac{dy_2}{dt} &= \frac{c_1 \alpha_2 \bar{R}_2 y_2}{1 + N + \alpha_2 y_1} - c_2 y_2 \\ \frac{\bar{R}_1}{dt} &= \beta_1 (N - \bar{R}_1) \\ \frac{\bar{R}_2}{dt} &= \beta_2 (N - \bar{R}_2)\end{aligned}\tag{2.2}$$

with $N(0) \geq 0$, $y_1(0) \geq 0$, $y_2(0) \geq 0$, $\bar{R}_1(0) \geq 0$ and $\bar{R}_2(0) \geq 0$.

To reduce the number of parameters in the Model-B (2.2), we use the following transformations:

$$x = N/k, \quad y = \alpha_2 y_1, \quad z = \alpha_1 y_2, \quad e_1 = \frac{\alpha_1 b_1}{r}, \quad e_2 = \frac{c_1 \alpha_2}{r}, \quad d_1 = \frac{b_2}{r}, \quad d_2 = \frac{c_2}{r},$$

$$g_1 = \frac{\alpha_1}{r \alpha_2}, \quad g_2 = \frac{\alpha_2}{r \alpha_1}, \quad t = \frac{t}{r}, \quad R_1 = k \bar{R}_1, \quad R_2 = k \bar{R}_2, \quad f_1 = \frac{\beta_1 k}{r}, \quad f_2 = \frac{\beta_2 k}{r},$$

then the Model-B (2.2) becomes:

Model-C:

$$\begin{aligned} \frac{dx}{dt} &= r^2 x(1-x) - \frac{g_1 xy}{1+kx+z} - \frac{g_2 xz}{1+kx+y} \\ \frac{dy}{dt} &= \frac{e_1 R_1 y}{1+kx+z} - d_1 y \\ \frac{dz}{dt} &= \frac{e_2 R_2 z}{1+kx+y} - d_2 z \\ \frac{dR_1}{dt} &= f_1(x - R_1) \\ \frac{dR_2}{dt} &= f_2(x - R_2) \end{aligned} \quad (2.3)$$

with $x(0) \geq 0$, $y(0) \geq 0$, $z(0) \geq 0$, $R_1(0) \geq 0$ and $R_2(0) \geq 0$.

3. Boundedness of the System:

Lemma 3.1. *The set, $\Omega = \{(N, y_1, y_2, \bar{R}_1, \bar{R}_2) \in R_+^5 : b_1 c_1 N(t) + c_1 y_1(t) + b_1 y_2(t) \leq \frac{b_1 c_1 r k}{\Phi_1}, \bar{R}_1 + \bar{R}_2 \leq \frac{k(\beta_1 + \beta_2)}{\Phi_2}\}$, where $\Phi_1 = \min\{b_1, c_1\}$ and $\Phi_2 = \min\{\beta_1, \beta_2\}$.*

Proof:

Let

$$w_1(t) = b_1 c_1 N(t) + c_1 y_1(t) + b_1 y_2(t).$$

By using the equations in (2.2), we get,

$$\frac{dw_1}{dt} + \Phi_1 w_1 \leq b_1 c_1 k r$$

where $\Phi_1 = \min\{b_1, c_1\}$ and $N \leq k$ (from the first equation of (2.2)). By comparison theorem, we get as $t \rightarrow \infty$,

$$w_1 \leq \frac{b_1 c_1 r k}{\Phi_1} =: m_1.$$

From equation (2.2), let

$$w_2 = \bar{R}_1 + \bar{R}_2$$

$$\frac{dw_2}{dt} + \Phi_2 w_2 \leq k(\beta_1 + \beta_2)$$

where $\Phi_2 = \min\{\beta_1, \beta_2\}$ and $N \leq k$ (from the equation). By comparison theorem, we get as $t \rightarrow \infty$,

$$w_2 \leq \frac{k(\beta_1 + \beta_2)}{\Phi_2} =: m_2.$$

Hence the system (2.2) is proved that the set Ω is bounded.

4. Analysis of the System (2.3):

4.1. The existence of the system

The system (2.3) has four equilibrium points:

- (i) $E_1 \equiv (1, 0, 0, k, k)$
- (ii) $E_2 \equiv \left(\frac{d_1}{k(e_1-d_1)}, \frac{r^2(1-x)(1+kx)}{g_1}, 0, kx, kx \right)$, where $e_1 > d_1$ and $1 > x$.
- (iii) $E_3 \equiv \left(\frac{d_2}{k(e_2-d_2)}, 0, \frac{r^2(1-x)(1+kx)}{g_2}, \frac{d_2}{e_2-d_2}, \frac{d_2}{e_2-d_2} \right)$, where $e_2 > d_2$ and $k > x$.
- (iv) $E_4 \equiv (x^*, y^*, z^*, R_1^*, R_2^*)$,

where, $x^* = \frac{\sqrt{b^2-4ac}-b}{2a}$, ($x^* > 0$), and $a = e_1 e_2$,
 $b = \frac{ke_2 g_1 d_1 (e_2 - d_2)}{d_2} + \frac{ke_1 g_2 d_2 (e_1 - d_1)}{d_1} - ke_1 e_2$, $c = -k(e_2 g_1 d_1 + e_1 g_2 d_2)$,
 $y^* = \frac{x^*(e_2 - d_2) - d_2}{d_2}$, $z^* = \frac{x^*(e_1 - d_1) - d_1}{d_1}$, $R_1^* = R_2^* = x^*$,
 where $x^*(e_1 - d_1) > d_1$ and $x^*(e_2 - d_2) > d_2$.

4.2. Discussion of local stability behavior

The discussion of local stability of all the equilibriums are evaluated by Jacobian matrix and their eigenvalues. The following results are obtained from equilibriums $E_1(1, 0, 0, k, k)$, $E_2 \equiv \left(\frac{d_1}{k(e_1-d_1)}, \frac{r^2(1-x)(1+kx)}{g_1}, 0, kx, kx \right)$, $E_3 \equiv \left(\frac{d_2}{k(e_2-d_2)}, 0, \frac{r^2(1-x)(1+kx)}{g_2}, \frac{d_2}{e_2-d_2}, \frac{d_2}{e_2-d_2} \right)$ and $E_4(x^*, y^*, z^*, R_1^*, R_2^*)$ respectively.

- (i) For E_1 equilibrium point, the eigenvalues are $\lambda_1 = -1$, $\lambda_2 = -f_1$, $\lambda_3 = -f_2$, $\lambda_4 = \frac{e_1 k}{1+k} - d_1$ and $\lambda_5 = \frac{e_2 k}{1+k} - d_2$ of Jacobian matrix's characteristic equation. It is noticed that E_1 is asymptotically stable at local level, when these conditions $e_1 k < d_1(1+k)$ and $e_2 k < d_2(1+k)$ hold.
- (ii) For E_2 equilibrium point, the two eigenvalues are $\lambda_1 = -f_2$ and $\lambda_2 = \frac{x e_2}{1+x+y} - d_2$ and the other three eigenvalues can be taken from the cubic equation:

$$\lambda^3 + Q_1 \lambda^2 + Q_2 \lambda + Q_3 = 0.$$

Here, $Q_1 = f_1 + \frac{2x}{k} + \frac{y g_1}{(1+x)^2} - 1$, $Q_2 = f_1 \left(\frac{y g_1}{(1+x)^2} + \frac{2x}{k} - 1 \right) - \frac{e_1 g_1 y x^2}{(1+x)^3}$, $Q_3 = \frac{x y e_1 g_1 f_1}{(1+x)^3}$. With Routh-Hurwitz criterion, it is observed that E_2 is stable at local level if $Q_1 > 0, Q_2 > 0, Q_3 > 0$ and $Q_1 Q_2 > Q_3$ hold.

Now we can state the following theorem for E_2 :

Theorem 4.1. *The point E_2 is locally asymptotically stable if*

$$\begin{aligned} (1+x)^2(kf_1+2x)+yg_1 &> k(1+x)^2 \\ f_1(1+x)(2x(1+x)^2+kyg_1) &> x^2ye_1g_1+f_1(1+x^3) \\ xe_2 &> d_2(1+x+y) \\ ((1+x)^2(f_1k+2x-k)+kyg_1)(m_1) &> xye_1g_1f_1k^2(1+x) \end{aligned} \tag{4.1}$$

where $m_1 = f_1(1+x)^2(2x-k)+kyg_1(f_1-e_1x^2)$.

(iii) For E_3 equilibrium point, the two eigenvalues are $\lambda_1 = -f_1$ and $\lambda_2 = \frac{xe_1}{1+x+z} - d_1$ and the other three eigenvalues can be taken from the cubic equation:

$$\lambda^3 + S_1\lambda^2 + S_2\lambda + S_3 = 0$$

Here, $S_1 = f_2 + \frac{2x}{k} + \frac{zg_2}{(1+x)^2} - 1$, $S_2 = f_2 \left(\frac{zg_2}{(1+x)^2} + \frac{2x}{k} - 1 \right) - \frac{e_2g_2zx^2}{(1+x)^3}$, $S_3 = \frac{xe_2g_2f_2}{(1+x)^3}$. With Routh-Hurwitz criterion, it is observed that E_2 is stable at local level if $S_1 > 0, S_2 > 0, S_3 > 0$ & $S_1S_2 > S_3$ hold. For E_3 , graphs are plotted to illustrate the behavior.

Now we can state the following theorem for E_3 :

Theorem 4.2. *The point E_3 is locally asymptotically stable if*

$$\begin{aligned} (1+x)^2(kf_2+2x)+zg_2 &> k(1+x)^2 \\ f_2(1+x)(2x(1+x)^2+kzg_2) &> x^2ze_2g_2+f_2(1+x^3) \\ xe_1 &> d_1(1+x+z) \\ ((1+x)^2(f_2k+2x-k)+kzg_2)(m_2) &> xye_1g_1f_1k^2(1+x) \end{aligned} \quad (4.2)$$

where $m_2 = f_2(1+x)^2(2x-k) + kzg_2(f_2 - e_2x^2)$.

Theorem 4.3. *The point E_4 is locally asymptotically stable if the following conditions hold.*

- (i) $d_1 + d_2 + f_1 + f_2 > J_1 1 + l_3 + l_4$
- (ii) $l_3 l_4 + d_1 d_2 + l_2 l_4 p_3 + f_1 f_2 > (f_1 + f_2)(l_3 + l_4 - d_1 - d_2 + J_1 1) + d_2 l_3 + l_4 d_1 + l_3 l_4 p_3 p_4 + l_1 l_3 p_4$
- (iii) $f_2 e_2 p_3 l_2 + f_1 e_1 p_4 l_1 > f_1 f_2 (l_3 + l_4 - d_1 - d_2 + J_1 1) + (f_1 + f_2)(l_3 l_4 (1 - p_3 p_4) - d_2 l_3 - l_4 d_1 + l_1 l_3 p_4 - l_2 l_4 p_3) + J_1 1 (l_3 l_4 (1 - p_3 p_4) - d_2 l_3 - l_4 d_1 + d_1 d_2 - l_3 - l_4 + d_1 + d_2) + l_1 l_3 p_4 (l_4 - d_2 + l_4 p_3) + l_2 l_4 p_3 (l_3 p_4 + l_3 - d_1)$
- (iv) $f_2 e_2 p_3 (l_2 d_2 + f_1 l_2 - l_1 l_3 p_4 - l_2 l_3) + f_1 e_1 p_4 (l_1 d_2 + f_2 l_1 - l_2 l_4 p_3 - l_1 l_4) + (f_1 + f_2)[J_{11} (l_3 l_4 (1 - p_3 p_4) + d_1 d_2 - d_2 l_3 - l_4 d_1) + l_1 l_3 p_4 (l_4 - d_2 + l_4 p_3) + l_2 l_4 p_3 (l_3 p_4 + l_3 - d_1)] > f_1 f_2 (J_1 1 (-l_3 - l_4 + d_1 + d_2) - d_1 d_2 + d_2 l_3 + l_4 d_1 + l_3 l_4 (1 - p_3 p_4) + l_1 l_3 p_4 - l_2 l_4 p_3)$
- (v) $f_1 f_2 e_2 p_3 (l_2 d_1 - l_1 l_3 p_4 - l_2 l_3) > f_1 f_2 e_1 p_4 (l_1 l_4 - l_1 d_2 + l_2 l_4 p_3) + f_1 f_2 (J_1 1 (l_3 l_4 (1 - p_3 p_4) + d_1 d_2 - d_2 l_3 - l_4 d_1) + l_1 l_3 p_4 (l_4 - d_2 + l_4 p_3) + l_2 l_4 p_3 (l_3 p_4 + l_3 - d_1))$

4.3. Discussion of global stability behavior

To discuss the global (nonlinear) stability of E_4 , we have considered and studied the Liapunov function, and the results are stated in the form of Theorem 4.4.

Theorem 4.4. *The point E_4 is globally asymptotically stable if*

$$\begin{aligned}
 b_2(1 + N + \alpha_1 y_2) &> a_1 b_1 R_1 \\
 c_2(1 + N + \alpha_2 y_1) &> a_2 c_1 R_2 \\
 r b_2(1 + N + \alpha_1 y_2)^2 - a_1 b_1 R_1(1 + N + \alpha_1 y_2) &> k a_1^2 \\
 r c_2(1 + N + \alpha_2 y_1)^2 - a_2 c_1 R_2(1 + N + \alpha_2 y_1) &> k a_2^2 \\
 r &> k \beta_1 \\
 r &> k \beta_2 \\
 b_2 \beta_1(1 + N + \alpha_1 y_2)^2 - a_1 b_1 \beta_1 R_1(1 + N + \alpha_1 y_2) &> (y_1 b_1 a_1)^2 \\
 c_2 \beta_2(1 + N + \alpha_2 y_1)^2 - a_2 c_1 \beta_2 R_2(1 + N + \alpha_2 y_1) &> (y_2 c_1 a_2)^2
 \end{aligned}$$

hold in the region Ω .

5. Numerical Simulation

In this section, we demonstrate the dynamical behavior of the Model-A and Model-B ((2.1) and (2.2)) with the help of numerical simulations. The purpose of numerical simulation is to interpret the mathematical results, and also to observe the effect of delay in the mathematical model (with delay and without delay). The figures illustrate the stability behavior of all the equilibrium points of the model and the graphs have been plotted with the help of MATLAB.

For the Model-B (2.2) with distributed delay:

- For E_1 equilibrium point, considered the set of the parameter values: $\alpha_1 = 1, c_1 = 0.4, c_2 = 0.12, \alpha_2 = 1, b_1 = 0.4, b_2 = 0.12, \beta_1 = 0.1, \beta_2 = 0.1, r = 0.2, k = 2, a_1 = 0.4, a_2 = 0.4$.

It has been found the equilibrium point E_1 as:

$$N = 2.0, y_1 = 0.0, y_2 = 0.0 \text{ is stable (see Fig.1).}$$

- For E_2 , considered the set of the parameter values: $b_2 = 0.1, c_2 = 0.1, \beta_1 = 0.05, \beta_2 = 0.02$.

With the above values of parameters and taking the remaining parameters to be the same as considered for E_1 , then it has been found the equilibrium point E_2 as:

$$N = 1.667, y_1 = 0.222, y_2 = 0.0 \text{ is stable (see Fig.2).}$$

- For E_3 , considered the set of the parameter values: $b_2 = 0.1, c_2 = 0.1, \beta_1 = 0.02, \beta_2 = 0.05$.

With the above values of parameters and taking the remaining parameters to be the same as considered for E_1 , then it has been found the equilibrium point E_3 as:

$$N = 1.667, y_1 = 0.0, y_2 = 0.222 \text{ is stable (see Fig.3).}$$

- For E_4 , considered the set of the parameter values: $b_2 = 0.05, c_2 = 0.05, \beta_1 = 0.5, \beta_2 = 0.5$.

With the above values of parameters and taking the remaining parameters to be the same as considered for E_1 , then it has been found the equilibrium point E_4 as:

$$N = 0.6085, y_1 = 0.3387, y_2 = 0.3387 \text{ is stable (see Fig.4 and 5).}$$

- For distinct values of β_1 and β_2 (distributed delay parameters) are shown in Fig. 6 and 7.

The behavior of distributed delay parameters have checked at defferent parameter values of b_1 and b_2 , which are shown in Fig. 8 and 9.

For the Model-A (2.1) without distributed delay:

- For E_2 (without delay), considered the set of the parameter values:
 $a_1 = 0.4, a_2 = 0.4, b_1 = 0.4, b_2 = 0.1, r = 0.2, \alpha_1 = 0.1, \alpha_2 = 1, k = 2,$
 $c_1 = 0.4, c_2 = 0.1.$

It has been found the equilibrium point E_2 as:

$$N = 1.667, y_1 = 0.222, y_2 = 0.0 \text{ is stable (see Fig.10).}$$

- For E_3 (without delay) equilibrium point, considered the set of values:
 $\alpha_1 = 1, \alpha_2 = 0.1.$

With the above values of parameters and taking the remaining parameters to be the same as considered for E_2 , then it has been found the equilibrium point E_3 as:

$$N = 1.667, y_1 = 0.0, y_2 = 0.222 \text{ is stable (see Fig.11).}$$

- For E_4 (without delay) equilibrium point, considered the set of values:
 $b_2 = 0.05, \alpha_1 = 1, c_2 = 0.05.$

With the above values of parameters and taking the remaining parameters to be the same as considered for E_2 , then it has been found the equilibrium point E_4 as:

$$N = 0.6085, y_1 = 0.3387, y_2 = 0.3387 \text{ is stable (see Fig.12).}$$

- For distinct values of α_1 and α_2 (interference parameters) are shown in Fig. 13 - 15.

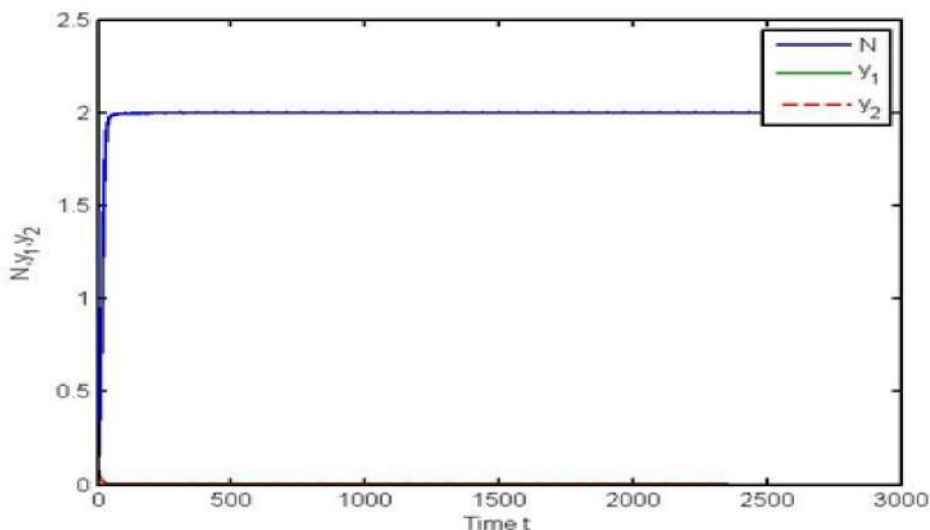


FIGURE 1. Stability behavior for Model-B (2.2) around the equilibrium point E_1 .

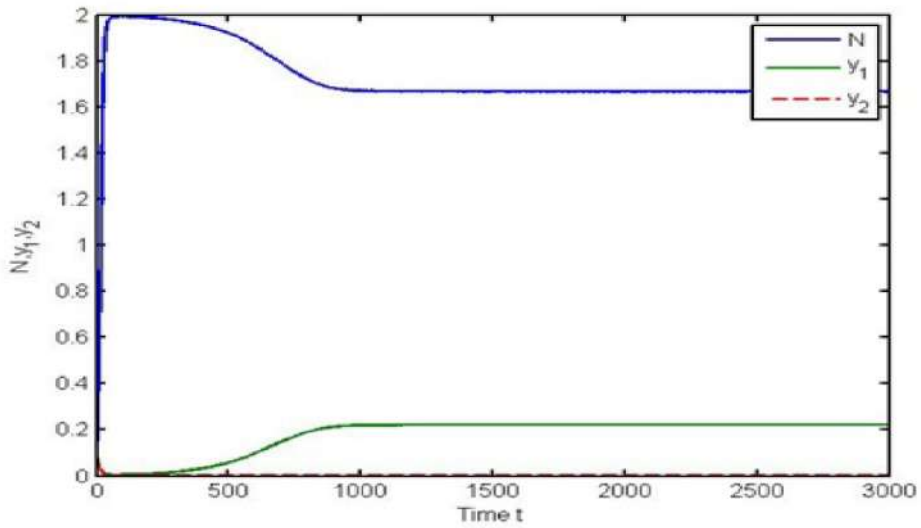


FIGURE 2. Stability behavior for Model-B (2.2) around the equilibrium point E_2 .

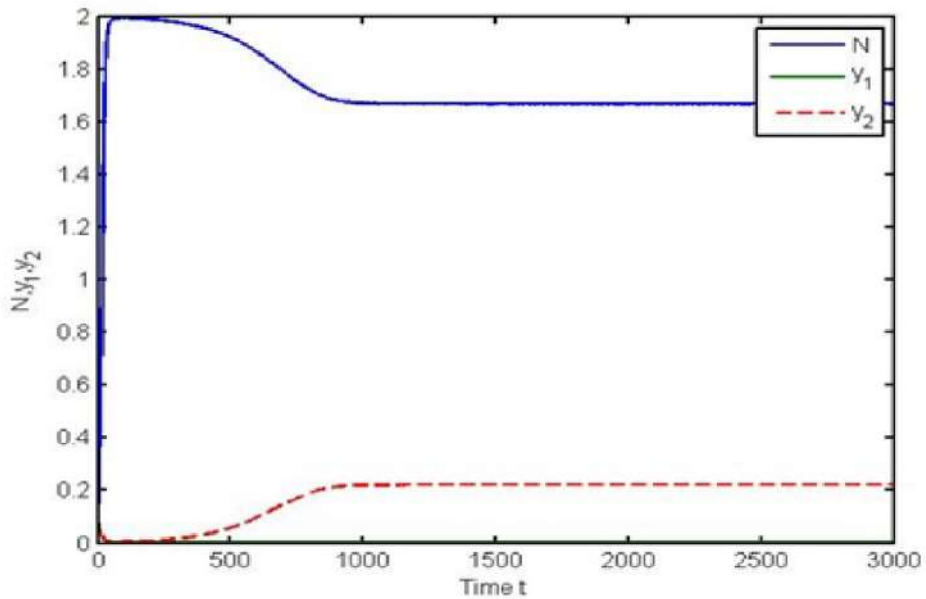


FIGURE 3. Stability behavior for Model-B (2.2) around the equilibrium point E_3 .

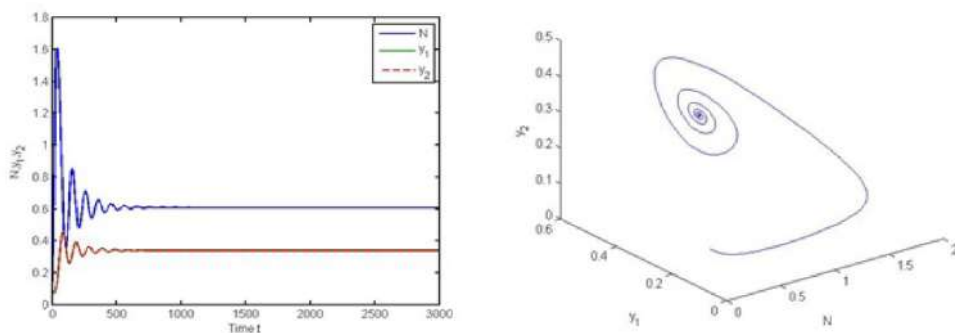


FIGURE 4. Stability behavior for Model-B (2.2) around the equilibrium point E_4 (when $\beta_1 = \beta_2 = 0.5$).

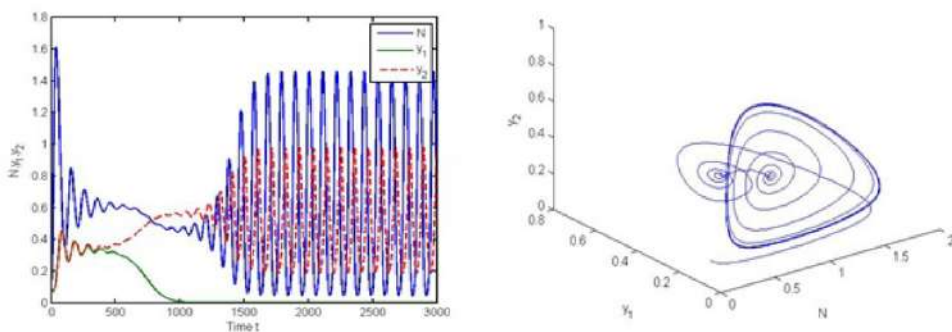


FIGURE 5. Time graph and phase diagram for the Model-B around the equilibrium point E_4 , showing unstable at $\beta_1 = 0.49$ and $\beta_2 = 0.5$.

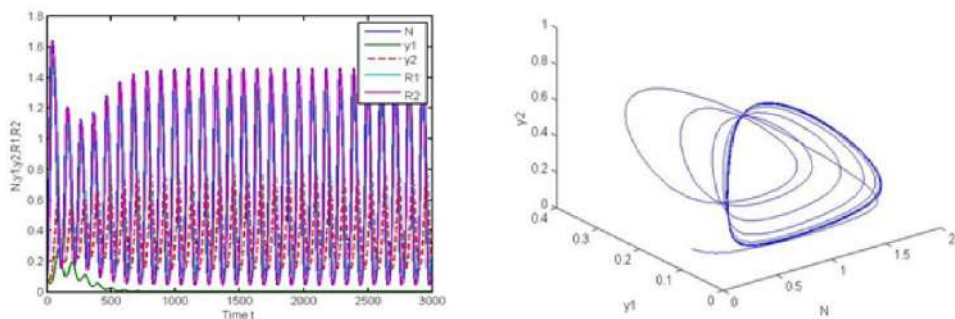


FIGURE 6. Time graph and phase diagram for the Model-B around the equilibrium point E_4 , showing unstable at $\beta_1 = 0.1$ and $\beta_2 = 0.5$.

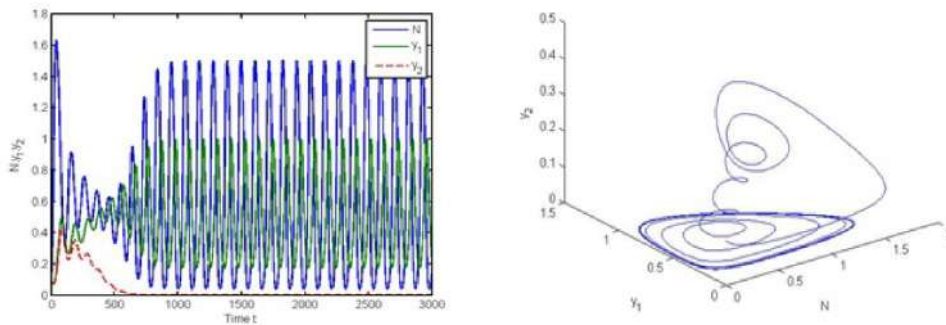


FIGURE 7. Time graph and phase diagram for the Model-B around the equilibrium point E_4 , showing unstable at $\beta_1 = 0.45$ and $\beta_2 = 0.3$.

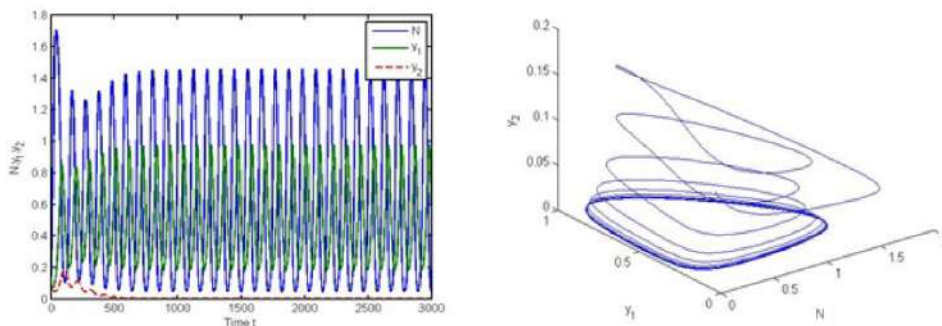


FIGURE 8. Time graph and phase diagram for the Model-B around the equilibrium point E_4 , showing unstable at $\beta_1 = 0.5$ and $\beta_2 = 0.05$.

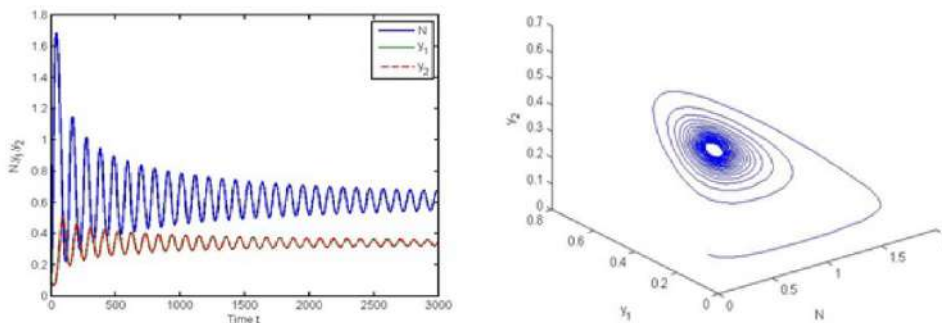


FIGURE 9. Time graph and phase diagram for the Model-B around the equilibrium point E_4 , showing unstable at $\beta_1 = \beta_2 = 0.21$.

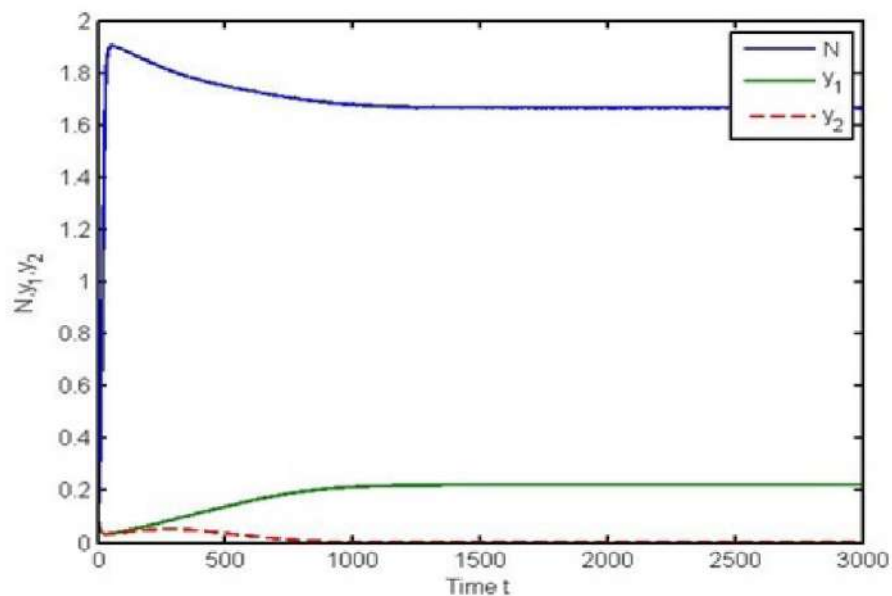


FIGURE 10. Stability behavior for Model-A (2.1) around the equilibrium point E_2 .

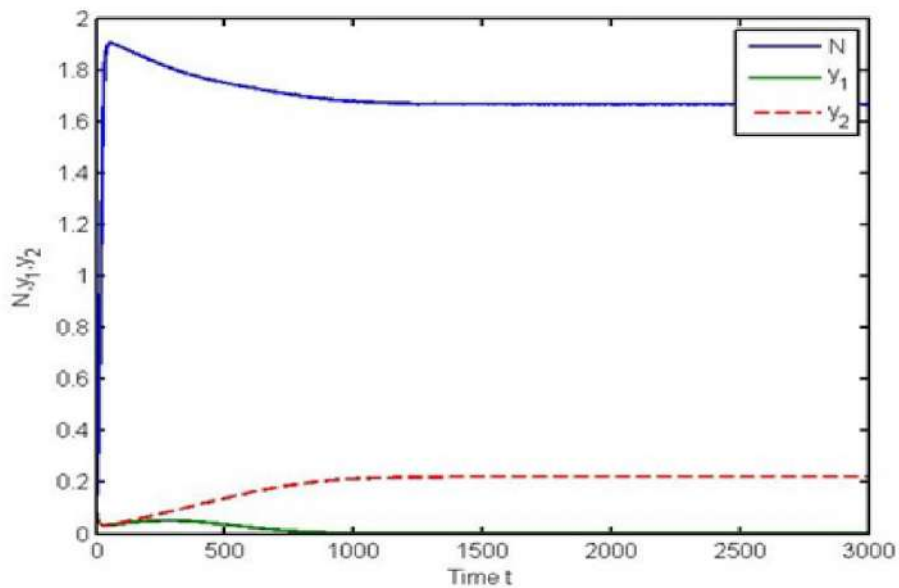


FIGURE 11. Stability behavior for Model-A (2.1) around the equilibrium point E_3 .

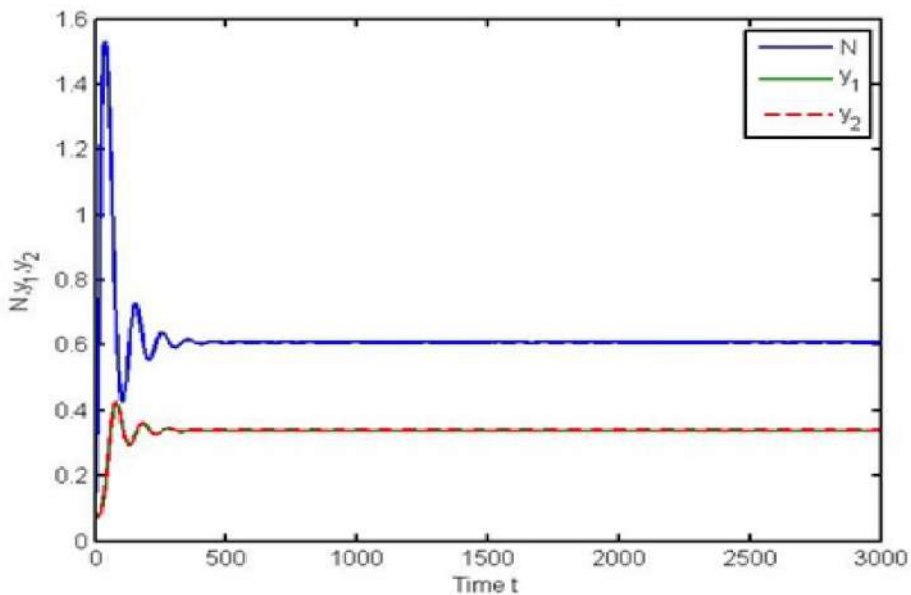


FIGURE 12. Stability behavior for Model-A (2.1) around the equilibrium point E_4 .

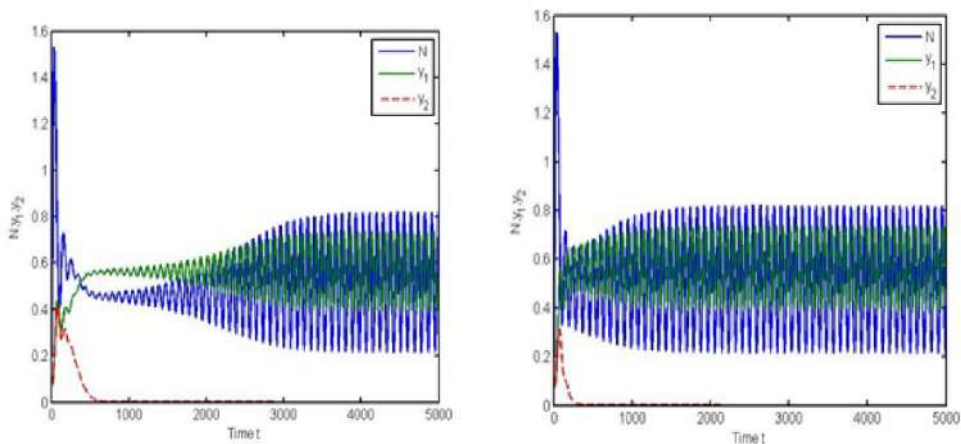


FIGURE 13. Time graph for the Model-A around the equilibrium point E_4 , showing unstable at (a) $\alpha_1 = 0.9, \alpha_2 = 1$ and (b) $\alpha_1 = 0.5, \alpha_2 = 1.5$.

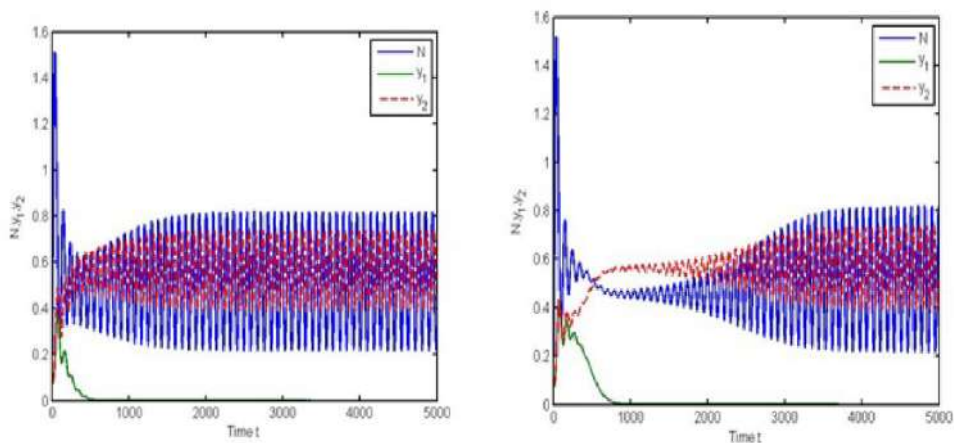


FIGURE 14. Time graph for the Model-A around the equilibrium point E_4 , showing unstable at (a) $\alpha_1 = 0.8, \alpha_2 = 0.3$ and (b) $\alpha_1 = 0.75, \alpha_2 = 0.7$.

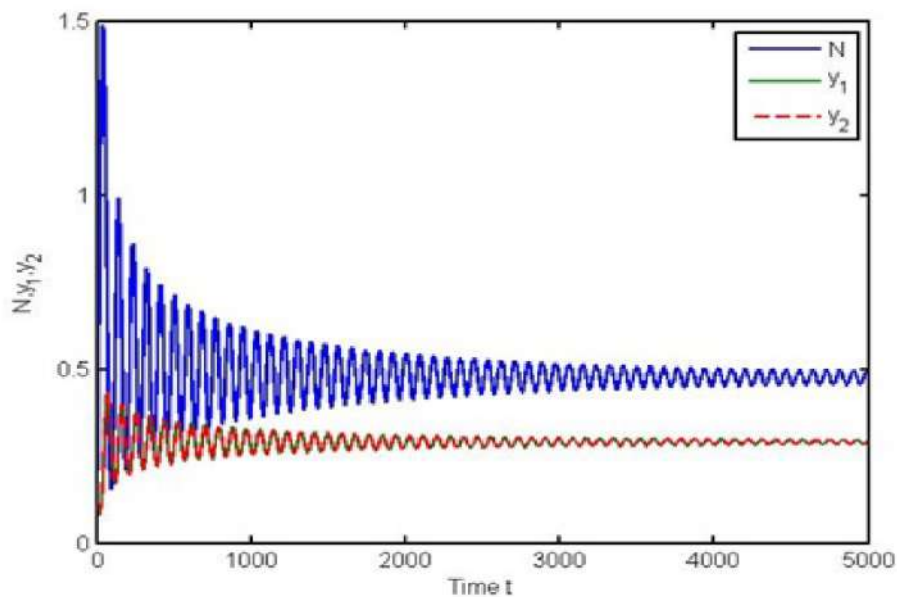


FIGURE 15. Time graph for the Model-A around the equilibrium point E_4 , showing stable at $\alpha_1 = \alpha_2 = 0.15$.

Table 1: Calculations of α_1 , α_2 , β_1 and β_2 for the Model-B (2.2) with distributed delay.

α_1	α_2	β_1	β_2	Remark
1	1	0.23	0.23	Unstable
1	1	0.24	0.24	Stable, $x=0.6085, y_1 = y_2 = 0.3386$
1	1	0.25	0.25	Stable, $x=0.6085, y_1 = y_2 = 0.3386$
1	1	0.5	0.5	Stable, $x=0.6085, y_1 = y_2 = 0.3386$
1	1	100	100	Stable, $x=0.6085, y_1 = y_2 = 0.3386$
0.57	0.57	0.5	0.5	Unstable
0.58	0.58	0.5	0.5	Stable, $x=0.5380, y_1 = y_2 = 0.3142$
4	4	0.5	0.5	Stable, $x=1.1626, y_1 = y_2 = 0.3894$
5	5	0.5	0.5	Stable, $x=1.2882, y_1 = y_2 = 0.3668$
5.05	5.05	0.5	0.5	Stable, $x=1.2936, y_1 = y_2 = 0.3655$
1000	1000	0.5	0.5	Stable, $x=1.9958, y_1 = y_2 = 0.0034$

Table 2: Calculations of α_1 and α_2 for the Model-A (2.1) without distributed delay.

α_1	α_2	Remark
0.8	0.8	Unstable
0.85	0.85	Stable, $x=0.5860, y_1 = y_2=0.3299$
1.1	1.1	Stable, $x=0.6260, y_1 = y_2=0.3440$
3	3	Stable, $x=1.0001, y_1 = y_2=0.4000$
3.1	3.1	Stable, $x=1.0182, y_1 = y_2=0.3999$
5	5	Stable, $x=1.2881, y_1 = y_2=0.3668$
100	100	Stable, $x=1.9574, y_1 = y_2=0.0331$
1000	1000	Stable, $x=1.9957, y_1 = y_2=0.0034$

6. Conclusion

Considered the distributed delay for the system of one prey and two competing predators, and incorporated the function of Beddington DeAngelis response. It's concluded from the stability of E_1 of (2.3), that only prey will live and predators would vanish (Fig.1). For the point E_2 and E_3 of the system (2.3), we can see the prey population and one of the predators population will live and the other predator population extinction (Figs. 2, 3). The point (E_4) of (2.3), is locally and globally showing the stability of all the population of prey and predator populations (Fig. 4).

Comparing the phase space graphs for different values of delay parameters (Fig. 4-9), it is observed that, all three species co-exist only when the delay parameters are equal, i.e., $\beta_1 = \beta_2$ for the range 0.24 and above, see Table 1 and Fig.9. Considering the set of parametric values for the E_4 , if $\beta_1 < \beta_2$ then the system becomes unstable (Fig.5 and 6) and if $\beta_1 > \beta_2$ then the system becomes unstable (Fig.7 and 8).

For the system without distributed delay, Figs. 10 - 12 show the stability of E_2, E_3 and E_4 respectively. Considering the set of parametric values considered for the equilibrium E_4 , if $\alpha_1 > \alpha_2$, then the system becomes unstable (Fig. 12) and if $\alpha_1 < \alpha_2$, then the

system is unstable (Fig. 11). Comparing the graphs for different values of delay parameters (Fig. 4-7), it is observed that, all the three species co-exist only when the delay parameters are equal, i.e., $\alpha_1 = \alpha_2$ for the range 0.85 and above, see Table 2 and Fig.13.

It is observed for the delay case (2.3) that β_1 and β_2 values take 0.24 and above, then the system will be stable and the values above 0.57 for α_1 and α_2 is stable. Also observed that as $\alpha_1, \alpha_2, \beta_1$ and β_2 increases then the prey population would be equal to the carrying capacity of the system, and the predator populations would extinction (see Table 1).

It is observed for without delay case (2.1) that α_1 and α_2 values take 0.85 and above then the system will be stable. Also observed that as α_1 and α_2 increase then the prey population would be equal to the carrying capacity of the system, and the predator populations would extinction (see Table 2).

It is noted that the density population of prey and predators in (2.1) (without delay case) is equal to the density population of prey and predators in (2.3) (with delay case). From the above note we can conclude in this case that delay does not affect the system (2.1) [25, 26, 31, 34] .

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