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# A Note on Laplace Adomian Decomposition Method on A Linear Problem

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**Abstract** In the last two decades, the Laplace Adomian Decomposition Method (LADM) has been vastly used to solve nonlinear (or even fractional order) differential equations. The method approaches the solution with the partial sums of function series. However, it is not easy to show that the limit of the function series is the exact solution to the problem. In this article, we consider a simple problem such as a homogenous second-order linear ordinary differential equation with constant coefficients. We prove analytically that the LADM gives the right exact solution to the considered problem.

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## 1. Introduction

In [5], Khuri presented the Laplace Adomian Decomposition Method (abbreviated as LADM) for solving nonlinear differential equations. The idea of the technique is to assume an infinite solution of the form $y = \sum_{n=0}^{\infty} y_n$, then apply the Laplace transformation to the problem. The nonlinear term is decomposed in terms of Adomian polynomials [1]. By ignoring the interchangeability of infinite sum and integral, an iterative algorithm is constructed for solving $u_n$ in a recursive manner.

In the last two decades, LADM has been widely considered to solve nonlinear differential equations. In 2009, Kiymaz used LADM to solve an initial value problem involving the Cauchy-Euler operator [6]. In 2010, LADM was considered by Tsai to approximate an analytic solution of nonlinear Riccati differential equations [7]. In 2012, Heris analyzed the convergence of LADM for integro-differential equations [4]. The LADM is also vastly applied to system differential equations (see [2, 3, 9]). Furthermore, LADM is also considered for fractional order problems (see [8, 10]).

Since the interchangeability of infinite sum and integral is ignored, we are curious whether the approximation solution by LADM gives the same results as the exact one.

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In this paper, we consider a simple problem such as the homogeneous second-order linear ordinary differential equations with a constant coefficient. We obtain the following result which we prove in the last section.

**Theorem 1.1.** *The LADM gives the exact solution to any homogeneous second-order linear ordinary differential equations with constant coefficient.*

## 2. Preliminaries

### 2.1. Laplace Adomian Decomposition Method

Let us recall the LADM from [5]. Let $f(y)$ be a nonlinear operator. We consider the homogeneous second-order nonlinear differential equation

$$y'' + ay' + by = f(y) \tag{2.1}$$

with initial-value $y(0) = \alpha$ and $y'(0) = \beta$. The idea of the LADM is to decompose the solution as an infinite series,

$$y = \sum_{n=0}^{\infty} y_n$$

and also to decompose the nonlinear operator $f(y)$ as

$$f(y) = \sum_{n=0}^{\infty} A_n,$$

where $A_n$ is the Adomian polynomials given by

$$A_n = \frac{1}{n!}\frac{\mathrm{d}^n}{\mathrm{d}u^n} f\left(\sum_{i=0}^{\infty} u^i y_i\right)\Bigg|_{u=0}$$

for each $n \in \mathbb{N} \cup \{0\}$. Thus, applying the Laplace transform (with notation $\mathcal{L}$) to (2.1), and applying the decomposition of $y$ and $f(y)$, we have

$$\mathcal{L}\left(\sum_{n=0}^{\infty} y_n\right) = \frac{\alpha}{s} + \frac{\beta + a\alpha}{s^2} - \frac{a}{s}\mathcal{L}\left(\sum_{n=0}^{\infty} y_n\right) - \frac{b}{s^2}\mathcal{L}\left(\sum_{n=0}^{\infty} y_n\right) + \frac{1}{s^2}\mathcal{L}\left(\sum_{n=0}^{\infty} A_n\right) \tag{2.2}$$

Notice that, we still do not know whether the integral (from the Laplace transform) and the infinite sum (from the decomposition of $y$ or $f(y)$) are interchangeable. Regardless, we pair both sides of (2.2) as follows:

$$\mathcal{L}(y_0) = \frac{\alpha}{s} + \frac{\beta + a\alpha}{s^2}$$
$$\mathcal{L}(y_n) = -\frac{a}{s}\mathcal{L}(y_{n-1}) - \frac{b}{s^2}\mathcal{L}(y_{n-1}) + \frac{1}{s^2}\mathcal{L}(A_{n-1}) \quad \text{for each } n \in \mathbb{N}.$$

Applying the inverse Laplace transform, we recursively solve for $y_n$, and obtain:

$$y_0(t) = \alpha + (\beta + a\alpha)t \tag{2.3}$$
$$y_n(t) = -a(1 * y_{n-1})(t) - b(1 * 1 * y_{n-1})(t) + (1 * 1 * A_{n-1})(t) \tag{2.4}$$

for each $n \in \mathbb{N}$, where $*$ is the convolution;

$$(f * g)(t) = \int_0^t f(\tau)g(t-\tau)\mathrm{d}\tau.$$

### 2.2. Homogenoeus second-order differential equation

Let us consider (2.1) with $f(y) = 0$, as follows

$$y'' + a\,y' + b\,y = 0, \quad y(0) = \alpha, \quad y'(0) = \beta. \tag{2.5}$$

Assume that $c$ and $d$ are complex numbers that satisfy $a = -(c+d)$ and $b = c\,d$. It is well known that the general solution of (2.5) are

$$y = A\,\mathrm{e}^{c\,t} + B\,\mathrm{e}^{d\,t}, \text{for } c \neq d,$$
$$y = A\,\mathrm{e}^{c\,t} + B\,t\,\mathrm{e}^{c\,t}, \text{for } c = d.$$

Given the initial value $y(0) = \alpha$ and $y'(0) = \beta$, we have the exact solution of (2.5) are

$$y = \frac{\beta - \alpha d}{d - c}\,\mathrm{e}^{c\,t} + \frac{\alpha c - \beta}{d - c}\,\mathrm{e}^{d\,t}, \text{for } c \neq d, \tag{2.6}$$
$$y = \alpha\,\mathrm{e}^{c\,t} + (\beta - c\alpha)\,t\,\mathrm{e}^{c\,t}, \text{for } c = d. \tag{2.7}$$

## 3. Proof of Theorem 1.1

Let us rewrite (2.3) and (2.4) in term of $c$ and $d$ as in Subsection 2.2. (Note that since $f(y) = 0$, we have $A_n = 0$ for all $n$.)

$$y_0(t) = \alpha + (\beta - (c+d)\alpha)t \tag{3.1}$$
$$y_n(t) = (c+d)(1 * y_{n-1})(t) - (cd)(1 * 1 * y_{n-1})(t) \tag{3.2}$$

for each $n \in \mathbb{N}$. Now let us consider separately the case of $c \neq d$ and the case of $c = d$.

### 3.1. The case of $c \neq d$

Since $c \neq d$, we can write (3.1) as

$$y_0(t) = \frac{\beta - \alpha\,d}{d - c}\,(1 - c\,t) + \frac{\alpha\,c - \beta}{d - c}\,(1 - d\,t) = A\sum_{j=1}^{2} c_{0,j} t^{j-1} + B\sum_{j=1}^{2} d_{0,j} t^{j-1}$$

with

$$A = \frac{\alpha\,d - \beta}{d - c}, \quad B = \frac{\beta - \alpha\,c}{d - c}, \quad c_{0,1} = d_{0,1} = 1, \quad c_{0,2} = -c \quad d_{0,2} = -d.$$

By (3.2), we expect that we have

$$y_n(t) = A\sum_{j=1}^{n+2} c_{n,j} t^{n+j-1} + B\sum_{j=1}^{n+2} d_{n,j} t^{n+j-1} \tag{3.3}$$

for each $n \in \mathbb{N} \cup \{0\}$. Suppose that (3.3) is true for every $n = k - 1$ with $k \in \mathbb{N}$, we aim to show that (3.3) is true for $n = k$. Since

$$y_k(t) = (c+d)\int_0^t y_{k-1}(\tau)\mathrm{d}\tau - cd\int_0^t \int_0^\tau y_{k-1}(w)\mathrm{d}w\mathrm{d}\tau,$$

and $y_{k-1}$ can be expressed as (3.3), we obtain

$$\begin{aligned} y_k(t) = A &\left[ \left( \frac{c+d}{k}\, c_{k-1,1} \right) t^k \right. \\ &+ \sum_{j=2}^{k+1} \left( \left( \frac{c+d}{k+j-1}\, c_{k-1,j} - \frac{c\,d}{(k+j-2)\,(k+j-1)}\, c_{k-1,j-1} \right) t^{k+j-1} \right) \\ &\left. - \left( \frac{c\,d}{2\,k\,(2\,k+1)}\, c_{k-1,k+1} \right) t^{2\,k+1} \right] + B \left[ \left( \frac{c+d}{k}\, d_{k-1,1} \right) t^k \right. \\ &+ \sum_{j=2}^{k+1} \left( \left( \frac{c+d}{k+j-1}\, d_{k-1,j} - \frac{c\,d}{(k+j-2)\,(k+j-1)}\, d_{k-1,j-1} \right) t^{k+j-1} \right) \\ &\left. - \left( \frac{c\,d}{2\,k\,(2\,k+1)}\, d_{k-1,k+1} \right) t^{2\,k+1} \right]. \end{aligned}$$

By setting

$$c_{k,j} = \begin{cases} \frac{c+d}{k}\, c_{k-1,1}, & \text{for } j = 1; \\ \frac{c+d}{k+j-1}\, c_{k-1,j} - \frac{c\,d}{(k+j-1)\,(k+j-2)}\, c_{k-1,j-1}, & \text{for } j = 2, 3 \ldots, k+1; \\ -\frac{c\,d}{2\,k\,(2\,k+1)}\, c_{k-1,k+1}, & \text{for } j = k+2, \end{cases} \tag{3.4}$$

and setting $d_{k,j}$ in the similar way as $c_{k,j}$, notice that $y_k$ can be expressed as (3.3). Thus, by induction (3.3) is valid for all $n \in \mathbb{N} \cup \{0\}$. Therefore, by the LADM, the approximation solution of (2.5) is given by

$$y(t) = \sum_{n=0}^{\infty} y_n(t) = A \sum_{n=0}^{\infty} \sum_{j=1}^{n+2} c_{n,j} t^{n+j-1} + B \sum_{n=0}^{\infty} \sum_{j=1}^{n+2} d_{n,j} t^{n+j-1}.$$

Now, we need to show that

$$\sum_{n=0}^{\infty} \sum_{j=1}^{n+2} c_{n,j} t^{n+j-1} = e^{ct}, \quad \text{and} \quad \sum_{n=0}^{\infty} \sum_{j=1}^{n+2} d_{n,j} t^{n+j-1} = e^{dt}.$$

Since $d_{n,j}$ is defined in similar way as $c_{n,j}$, it suffices to work on the summation of $c_{n,j}$. By the Maclaurin series

$$e^{ct} = \sum_{i=0}^{\infty} \frac{c^i}{i!} t^i,$$

we only need to show that for each $i \in \mathbb{N} \cup \{0\}$

$$\sum_{(n,j) \in C_i} c_{n,j} = \frac{c^i}{i!}, \tag{3.5}$$

where $C_i = \{(n, j) : n + j - 1 = i, \quad j \in \{1, \ldots, n + 2\}\}$. Let us first check for $i = 0$, $i = 1$, and $i = 2$. For $i = 0$, we have $\sum_{(n,j) \in C_0} c_{n,j} = c_{0,1} = 1$. For $i = 1$, by (3.4) $\sum_{(n,j) \in C_1} c_{n,j} = c_{0,2} + c_{1,1} = -d + (c + d) c_{0,1} = c$. Lastly, for $i = 2$, and by (3.4) we have

$$\sum_{(n,j) \in C_2} c_{n,j} = c_{1,2} + c_{2,1} = \frac{(c+d)}{2} c_{0,2} - \frac{cd}{2} c_{0,1} + \frac{(c+d)^2}{2} c_{0,1} = \frac{c^2}{2}.$$

Assume that (3.5) holds for $i = m-2$ and $i = m-1$ with $m \geq 2$. It is now suffices to show that (3.5) is true for $i = m$. Note that, if $m$ is odd,

$$\sum_{(n,j)\in C_m} c_{n,j} = \sum_{j=1}^{\frac{m+1}{2}+1} c_{m+1-j,j}$$

and if $m$ is even,

$$\sum_{(n,j)\in C_m} c_{n,j} = \sum_{j=1}^{\frac{m}{2}+1} c_{m+1-j,j}.$$

For either case of $m$, by (3.4), we obtain

$$\begin{aligned}\sum_{(n,j)\in C_m} c_{n,j} &= \frac{c+d}{m} \sum_{(n,j)\in C_{m-1}} c_{n,j} - \frac{cd}{m(m-1)} \sum_{(n,j)\in C_{m-2}} c_{n,j} \\ &= \frac{c+d}{m}\frac{c^{m-1}}{(m-1)!} - \frac{cd}{m(m-1)}\frac{c^{m-2}}{(m-2)!} = \frac{c^m}{m!}.\end{aligned}$$

Hence, by induction, (3.5) holds for each $i \in \mathbb{N} \cup \{0\}$. Thus, we can conclude that for the case of $c \neq d$, LADM gives the same solutions as (2.6).

### 3.2. The case of $c = d$

Note that (3.1) can be rewritten as

$$y_0(t) = \alpha\ (1 - c\,t) + (\beta - c\alpha)t = A\sum_{j=1}^{2} c_{0,j}t^{j-1} + Bt\sum_{j=1}^{1} d_{n,j}t^{j-1}$$

with

$$A = \alpha, \quad B = \beta - c\alpha, \quad c_{0,1} = d_{0,1} = 1, \quad c_{0,2} = -c.$$

By (3.2), we expect that

$$y_n(t) = A\sum_{j=1}^{n+2} c_{n,j}t^{n+j-1} + Bt\sum_{j=1}^{n+1} d_{0,j}t^{n+j-1} \tag{3.6}$$

for each $n \in \mathbb{N} \cup \{0\}$. Assume that (3.6) holds for $n = k-1$ with $k \in \mathbb{N}$, we shall prove that (3.6) is also valid for $n = k$. Since

$$y_k(t) = 2c\int_0^t y_{k-1}(\tau)\mathrm{d}\tau - c^2\int_0^t\int_0^\tau y_{k-1}(w)\mathrm{d}w\mathrm{d}\tau,$$

we obtain

$$
\begin{aligned}
y_k(t) = A \Bigg[ & \left(\frac{2\,c}{k} c_{k-1,1}\right) t^k \\
& + \sum_{j=2}^{k+1} \left( \left( \frac{2\,c}{k+j-1} c_{k-1,j} - \frac{c^2}{(k+j-2)\,(k+j-1)} c_{k-1,j-1} \right) t^{k+j-1} \right) \\
& - \left( \frac{c^2}{2\,k\,(2\,k+1)} c_{k-1,k+1} \right) t^{2\,k+1} \Bigg] + Bt \Bigg[ \left( \frac{2c}{k+1} d_{k-1,1} \right) t^k \\
& + \sum_{j=2}^{k} \left[ \left( \frac{2\,c}{k+j} d_{k-1,j} - \frac{c^2}{(k+j-1)\,(k+j)} d_{k-1,j-1} \right) t^{k+j-1} \right] \\
& - \frac{c^2}{2\,k\,(2\,k+1)} d_{k-1,k} t^{2\,k} \Bigg].
\end{aligned}
$$

By letting $c_{k,j}$ be the same as (3.4), and setting

$$
d_{k,j} = \begin{cases} \frac{2\,c}{k+1}\, d_{k-1,1}, & \text{for } j=1; \\ \frac{2\,c}{k+j}\, d_{k-1,j} - \frac{c^2}{(k+j-1)\,(k+j)}\, d_{k-1,j-1}, & \text{for } j=2,3\ldots,k; \\ -\frac{c^2}{2\,k\,(2\,k+1)}\, d_{k-1,k}, & \text{for } j=k+1, \end{cases} \tag{3.7}
$$

then $y_k$ can be written as (3.6). Therefore, (3.6) is valid for any $n \in \mathbb{N} \cup \{0\}$ by induction. Hence, by the LADM, the approximation solution of (2.5) is given by

$$
y(t) = \sum_{n=0}^{\infty} y_n(t) = A \sum_{n=0}^{\infty} \sum_{j=1}^{n+2} c_{n,j} t^{n+j-1} + Bt \sum_{n=0}^{\infty} \sum_{j=1}^{n+1} d_{n,j} t^{n+j-1}.
$$

In Subsection 3.1, we have shown that

$$
\sum_{n=0}^{\infty} \sum_{j=1}^{n+2} c_{n,j} t^{n+j-1} = e^{ct}.
$$

We now need to prove that

$$
\sum_{n=0}^{\infty} \sum_{j=1}^{n+1} d_{n,j} t^{n+j-1} = e^{ct}.
$$

Thus, we verify that for each $i \in \mathbb{N} \cup \{0\}$

$$
\sum_{(n,j)\in D_i} d_{n,j} = \frac{c^i}{i!} \tag{3.8}
$$

where $D_i = \{(n,j) : n+j-1 = i,\ \ j \in \{1,\ldots,n+1\}\}$. Let us first verify for $i \in \{0,1,2\}$. For $i = 0$, we have $\sum_{(n,j)\in D_0} d_{n,j} = d_{0,1} = 1$. Next, for $i = 1$, by (3.7) we can calculate $\sum_{(n,j)\in D_1} d_{n,j} = d_{1,1} = c d_{0,1} = c$. Then, for $i = 2$, by (3.7), we have

$$
\sum_{(n,j)\in D_2} d_{n,j} = d_{1,2} + d_{2,1} = -\frac{c^2}{3!} d_{0,1} + \frac{4c^2}{3!} d_{0,1} = \frac{c^2}{2}.
$$

Suppose that (3.8) valid for $i = m-2$ and $i = m-1$ with $m \geq 2$. We shall show that (3.8) holds for $i = m$. If $m$ is odd, then

$$\sum_{(n,j)\in D_m} d_{n,j} = \sum_{j=1}^{\frac{m+1}{2}} d_{m+1-j,j}.$$

For even $m$, we have

$$\sum_{(n,j)\in D_m} d_{n,j} = \sum_{j=1}^{\frac{m}{2}+1} d_{m+1-j,j}.$$

For either case of $m$, by (3.7)

$$\begin{aligned}\sum_{(n,j)\in D_m} d_{n,j} &= \frac{2c}{m+1}\sum_{(n,j)\in D_{m-1}} d_{n,j} - \frac{c^2}{m(m+1)}\sum_{(n,j)\in D_{m-2}} d_{n,j}\\ &= \frac{c^m}{(m+1)(m-2)!}\left(\frac{2}{m-1} - \frac{1}{m}\right) = \frac{c^m}{m!}.\end{aligned}$$

Thus, by induction (3.8) holds for any $i \in \mathbb{N} \cup \{0\}$. This verifies that for the case of $c = d$, LADM gives the same solutions as the exact one (2.7). This completes the proof of Theorem 1.1.

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