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$\log -$Series and $\log -$functions as application of multidual analysis

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Abstract The purpose of this paper is to contribute to the development of the new concept of $\log -$series and $\log -$functions as a particular continuation of real power series and real functions in multidual Algebra. We will focus on the case of elementary $\log -$functions and we propose some applications for special functions.

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1. Introduction

Multidual numbers have been introduced for the first time by F. Messelmi in the reference [16] as the generalization of dual numbers in higher dimensions. The idea is to consider a unit number satisfying $\varepsilon^{n+1} = 0$ and create the $(n+1)-$dimensional associative, commutative, and unitary generalized Clifford algebra generated by ε, said to be multidual algebra. The author studied the function of multidual variables. In detail, the formulas of Cauchy-Riemann were generalized and some results regarding the continuation of multidual functions have been also shown. Moreover, in the reference [12] the author has generalized multidual numbers over the set of complex numbers by introducing the concept of multidual complex numbers and he has studied the multidual complex functions and their inverses. In-depth, algebraic properties of the multidual algebra were elaborated in the refernces [16–18]. Differential calculus of multidual functions was the subject of the paper [15]. Indeed, the author introduces the notions of anti-hyperholomorphic functions and co-hyperholomorphic functions as well as the concept of generalized Dirac operators and he established many interesting results. Furthermore, multidual analysis have found various applications in technological fields, particularly in Mechanics, Robotics, Aeronotics and Electronics, see for more details the references [3–11].

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The main purpose of the present paper is to introduce the concept of log −series and log −functions. The paper is organized as follows. We will focus in the second section on recalling the basic properties of multidual analysis, notably hyperholomorphic functions, see the reference [16]. The third section aims to generalize the factorial for multidual integers and we define the new concepts of log −series and log −functions. We will restrict ourselves in this work to the study of the elementary log −functions that represent a particular extension of the classical elementary real functions. We show in addition that we can use the elementary log −functions to provide the expansion of some special functions written as integrals involving the $n-$th power of logarithmic function making use of harmonic numbers.

2. Preliminaries

A multidual number z is defined according to work of F. Messelmi [16] as an ordered $(n+1)-$tuple of real numbers $(x_0, x_1, ..., x_n)$ associated with the real unit 1 and the powers of the multidual unit ε, where ε is an $(n+1)-$nilpotent number i.e. $\varepsilon^{n+1} = 0$ and $\varepsilon^i \neq 0$ for $i = 1, ..., n$. Indeed, a multidual number is usually denoted in the form

$$z = \sum_{i=0}^{n} x_i \varepsilon^i.$$

for which, we admit that $\varepsilon^0 = 1$. The set of multidual numbers is denonted by $\mathbb{D}_n$ and given by

$$\mathbb{D}_n = \left\{ z = \sum_{i=0}^{n} x_i \varepsilon^i \mid x_i \in \mathbb{R} \text{ where } \varepsilon^{n+1} = 0 \text{ and } \varepsilon^i \neq 0 \text{ for } i = 1, ..., n \right\}.$$

If $z = \sum\limits_{i=0}^{n} x_i \varepsilon^i$ is a multidual number, we will denote by $\mathrm{real}\,(z)$ the real part of z given by

$$\mathrm{real}\,(z) = x_0.$$

The multidual numbers form a commutative ring with characteristic 0. Moreover, the inherited multiplication gives the multidual numbers the structure of $(n+1)-$dimensional generalized Clifford Algebra. For $n = 1$, $\mathbb{D}_1$ represents the Clifford algebra of dual numbers, see for more details regarding dual numbers the references [2, 12, 14]. In abstract algebra terms, the multidual ring can be obtained as the quotient of the polynomial ring $\mathbb{R}[X]$ by the ideal generated by the polynomial X^{n+1}, i.e.

$$\mathbb{D}_n \simeq \frac{\mathbb{R}[X]}{\langle X^{n+1} \rangle}.$$

It is also important to point out that every multidual number possess a matrix representation that can be formulated as follows. Let us denote by $\mathcal{G}_{n+1}(\mathbb{R})$ the subset of $\mathcal{M}_{n+1}(\mathbb{R})$ given by

$$\begin{aligned}\mathcal{G}_{n+1}(\mathbb{R}) = \{ A = (x_{ij}) \in \mathcal{M}_{n+1}(\mathbb{R}) \mid x_{ij} = 0 \text{ if } i < j \\ \text{and } x_{i+1,j+1} = x_{ij} \text{ if } j \leq i \leq n \}.\end{aligned}$$

So, an element A of $\mathcal{G}_{n+1}(\mathbb{R})$ can be written as

$$A=\begin{bmatrix} a_0 & 0 & \dots & 0 \\ a_1 & a_0 & \dots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ a_n & \dots & a_1 & a_0 \end{bmatrix}.$$

It is clear that $\mathcal{G}_{n+1}(\mathbb{R})$ is a subring of $\mathcal{M}_{n+1}(\mathbb{R})$ and it has also a structure of $(n+1)-$ dimensional associative, commutative, and unitary algebra. If $a_0 \neq 0$, $\mathcal{G}_{n+1}$ becomes a field. In particular, the set $\mathcal{G}_{n+1}(\mathbb{R})$ can be also seen as a subgroup of $GL(n+1)$.

There are many ways to choose the multidual unit number ε. The basic example can be given by the matrix

$$\varepsilon=\begin{bmatrix} 0 & 0 & \dots & 0 \\ 1 & 0 & \ddots & 0 \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 1 & 0 \end{bmatrix}.$$

Introducing now the following mapping

$$\begin{cases} \mathcal{R}:\mathbb{D}_n \longrightarrow \mathcal{G}_{n+1}(\mathbb{R}), \\ \mathcal{R}\left(\sum\limits_{i=0}^{n} x_i\varepsilon^i\right)=A=\begin{bmatrix} x_0 & 0 & \dots & 0 \\ x_1 & x_0 & \dots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ x_n & \dots & x_1 & x_0 \end{bmatrix}. \end{cases}$$

The result below shows the relationship between the sets $\mathbb{D}_n$ and $\mathcal{G}_{n+1}(\mathbb{R})$, [16].

Proposition 2.1. *$\mathcal{R}$ is an isomorphism of algebras.*

If z is a multidual number, the conjugate of z, denoted by $\bar{z}$, is the multidual number given by

$$z\bar{z}=\det\mathcal{R}(z)=(\mathrm{real}(z))^{n+1}.$$

Hence, $z=\sum\limits_{i=0}^{n} x_i\varepsilon^i$ has a unique conjugate if and only if $\mathrm{real}(z)=x_0\neq 0$. If $x_0=0$, the number $\sum\limits_{i=1}^{n} x_i\varepsilon^i$ is a divisor of zero in the ring $\mathbb{D}_n$. Denote by D the set of zero divisors of the ring $\mathbb{D}_n$, i.e.

$$D=\left\{\sum_{i=1}^{n} x_i\varepsilon^i \mid x_i\in\mathbb{R}\right\}.$$

For the sequel we admit that $\mathbb{D}_n$ is endowed with the usual topology of $\mathbb{R}^{n+1}$. We recall now, according to the work [16], some concepts and results regarding multidual functions.

Let Ω be an open subset of $\mathbb{D}_n$, $z=\sum\limits_{i=0}^{n} x_i\varepsilon^i\in\Omega$ and $f:\Omega\longrightarrow\mathbb{D}_n$ a multidual function. The Cauchy-Riemann conditions can be generalized for multidual function as follows.

Proposition 2.2. *Let f be a multidual function in $\Omega \subset \mathbb{D}_n$, which can be written in terms of its real and multidual parts as*

$$f(z) = \sum_{i=0}^{n} f_i(x_0, x_1, \dots, x_n) \varepsilon^i.$$

and suppose that the partial derivatives of f exist. Then,

1. *f is hyperholomorphic in Ω if and only if the following formulas hold*

$$\begin{cases} \dfrac{\partial f_i}{\partial x_j} = \dfrac{\partial f_{i-j}}{\partial x_0} & \text{if } j \leq i, \\ \dfrac{\partial f_i}{\partial x_j} = 0 & \text{if } j > i. \end{cases}$$

2. *f is hyperholomorphic in Ω if and only if its partial derivatives satisfy*

$$\frac{\partial f}{\partial x_j} = \varepsilon^j \frac{\partial f}{\partial x_0}, \qquad j = 0, \dots, n.$$

This allows us to deduce in particluar that if the function f is hyperholomorphic then

$$\frac{df}{dz} = \frac{\partial f}{\partial x_0}.$$

A multidual function defined in $\Omega \subset \mathbb{D}_n$ is said to be homogeneous if

$$f(\mathrm{real}(z)) \in \mathbb{R}.$$

The following proposition ensures that every regular real function can be extended to the algebra of multidual numbers.

Proposition 2.3 (Continuation of real functions)**.** *Let $f : O \longrightarrow \mathbb{R}$ be a real function, where O is an open connected domain of $\mathbb{R}$.*

1. *Suppose that $f \in \mathcal{C}^{n+1}(O)$. Then, there exists a unique homogeneous hyperholomorphic multidual function $\widetilde{f} : \Omega_O \subset \mathbb{D}_n \longrightarrow \mathbb{D}_n$ satisfying*

$$\widetilde{f}(x_0) = f(x_0) \quad \forall x_0 \in O,$$

where

$$\Omega_O = \left\{ z = \sum_{i=1}^{n} x_i \varepsilon^i \in \mathbb{D}_n \mid x_0 \in O \right\}.$$

2. *For $i = 1, \dots, n$ and $j = 1, \dots, i$, there exists real polynomials $P_{ij} \in \mathbb{R}[x_1, \dots, x_i]$ where $deg(P_{ij}) \leq i$, such that*

$$\widetilde{f}(z) = f(x_0) + \sum_{i=1}^{n} \sum_{j=1}^{i} P_{ij}(x_1, \dots, x_i) f^{(i-j+1)}(x_0) \varepsilon^i. \tag{2.1}$$

If in addition $f \in \mathcal{C}^q(O)$, $q \geq n+1$, then $\widetilde{f} \in \mathcal{C}^{q-n-1}(\Omega_O)$. In Particular, if $f \in \mathcal{C}^\infty(O)$, then $\widetilde{f} \in \mathcal{C}^\infty(\Omega_O)$, we say in such case that f is an analytic function in Ω_O.

In the following proposition, we give some properties regarding the generator polynomials P_{ij} appearing in formula (2.1).

Proposition 2.4. *The generator polynomials verify the following statements:*

$$\begin{cases} P_{ij} = 0 \quad \forall i = 1, \ldots, n \text{ and } j = i+1, \ldots, n, \\ \frac{\partial P_{ij}}{\partial x_k} = 0 \quad \forall i = 1, \ldots, n, \ k = 1, \ldots, i \text{ and } j = 1, \ldots, k-1, \\ \frac{\partial P_{ij}}{\partial x_k} = P_{i-k, j-k+1} \quad \forall i = 2, \ldots, n, \ k = 1, \ldots, i-1 \text{ and } j = k, \ldots, i-1, \\ P_{ii}(x_1, \ldots, x_i) = x_i \quad \forall i = 1, \ldots, n. \end{cases}$$

Furthermore, according to the work [18], the set of multidual integers $\mathbb{Z}_n(\varepsilon)$ is defined as

$$\mathbb{Z}_n(\varepsilon) = \left\{ m = \sum_{i=0}^{n} m_i \varepsilon^i \mid m_i \in \mathbb{Z} \right\}.$$

The set $\mathbb{Z}_n(\varepsilon)$ can be seen as a generated $\mathbb{Z}-$module having $(1, \varepsilon, \ldots, \varepsilon^n)$ as a system of generators. It is also important to note that $\mathbb{Z}_n(\varepsilon)$ can be also obtained as the quotient of the polynomial ring $\mathbb{Z}[X]$ by the ideal generated by the polynomial X^{n+1}, i.e.

$$\mathbb{Z}_n(\varepsilon) \simeq \frac{\mathbb{Z}[X]}{\langle X^{n+1} \rangle}.$$

The set of the zero divisors of the ring $\mathbb{Z}_n(\varepsilon)$ denoted by $D_n(\varepsilon)$ coincides with the ideal generated by ε. This means that,

$$D_n(\varepsilon) = \varepsilon \mathbb{Z}_n(\varepsilon) = \left\{ m = \sum_{i=1}^{n} m_i \varepsilon^i \right\}.$$

A multidual integer $m = \sum\limits_{i=1}^{n} m_i \varepsilon^i$ is said to be positive if $m_0 > 0$. The set of positive multidual integers is given by

$$\mathbb{Z}_n^+(\varepsilon) = \left\{ m = \sum_{i=0}^{n} m_i \varepsilon^i \in \mathbb{Z}_n(\varepsilon) \mid m_0 > 0 \right\}, \tag{2.2}$$

forms a commutative monoid under multiplication. Furthermore, for every $x \in \mathbb{R} - \mathbb{Z}^-$, we define the $p-$th generalized harmonic number, denoted by $H_{p,q}(x)$, see [1], by

$$H_{p,q}(x) = \sum_{r=0}^{p} \frac{1}{(x+r)^q}.$$

In order to simplify the notations, we will write from now on $H_{p,q} = H_{p,q}(1)$.

3. $\log -$Series, $\log -$Functions, and Applications

We start this section by suggesting in generalization of the factorial map for positive multidual integers. To do this, let us introduce the following definition.

Definition 3.1. The factorial of the integer $m = \sum\limits_{i=0}^{n} m_i \varepsilon^i \in \mathbb{Z}_n^+(\varepsilon)$ is defined by the formula

$$m! = \prod_{r=1}^{m_0} \left(r + \sum_{i=1}^{n} m_i \varepsilon^i \right). \tag{3.1}$$

In the following proposition we give an expression of the multidual factorial using the generator polynomials.

Proposition 3.2. *Let* $m=\sum\limits_{i=0}^{n}m_i\varepsilon^i\in\mathbb{Z}_n^+(\varepsilon)$. *We have*

$$m!=m_0!\left(1+\sum_{i=1}^{n}\sum_{j=1}^{i}P_{ij}\left(Y_1(m_0),\ldots,Y_i(m_0)\right)\varepsilon^i\right), \tag{3.2}$$

where

$$Y_i(m_0)=\sum_{r=1}^{m_0}\sum_{j=1}^{i}(-1)^{i-j}(i-j)!P_{ij}\left(\frac{m_1}{r},\ldots,\frac{m_i}{r}\right). \tag{3.3}$$

Proof. Let $m=\sum\limits_{i=0}^{n}m_i\varepsilon^i\in\mathbb{Z}_n^+(\varepsilon)$, in view of (3.1) we can write

$$\begin{aligned}m!&=m_0!\prod_{r=1}^{m_0}\left(1+\sum_{i=1}^{n}\frac{m_i}{r}\varepsilon^i\right)\\&=m_0!\prod_{r=1}^{m_0}e^{\sum\limits_{i=1}^{n}y_{ri}\varepsilon^i},\end{aligned}$$

where

$$e^{\sum\limits_{i=1}^{n}y_{ri}\varepsilon^i}=1+\sum_{i=1}^{n}\frac{m_i}{r}\varepsilon^i.$$

Thus

$$m!=m_0!e^{\sum\limits_{r=1}^{m_0}\sum\limits_{i=1}^{n}y_{ri}\varepsilon^i}.$$

Let us denote by $Y_i(m_0)$ the sum $Y_i(m_0)=\sum\limits_{r=1}^{m_0}y_{ri}$, one finds

$$m!=m_0!\left(1+\sum_{i=1}^{n}\sum_{j=1}^{i}P_{ij}\left(Y_1(m_0),\ldots,Y_i(m_0)\right)\varepsilon^i\right).$$

Moreover, it is clear that

$$\sum_{i=1}^{n}Y_i(m_0)\varepsilon^i=\sum_{r=1}^{m_0}\log\left(1+\sum_{i=1}^{n}\frac{m_i}{r}\varepsilon^i\right).$$

We can deduce, thanks to Proposition 2.3, that

$$\sum_{i=1}^{n}Y_i(m_0)\varepsilon^i=\sum_{r=1}^{m_0}\sum_{i=1}^{n}\sum_{j=1}^{i}P_{ij}\left(\frac{m_1}{r},\ldots,\frac{m_i}{r}\right)\log^{(i-j+1)}(1)\varepsilon^i.$$

Consequently

$$Y_i(m_0)=\sum_{r=1}^{m_0}\sum_{j=1}^{i}(-1)^{i-j}(i-j)!P_{ij}\left(\frac{m_1}{r},\ldots,\frac{m_i}{r}\right).$$

This achieves us the proof. ∎

Our goal now is to introduce the new concept of log −series.

Definition 3.3. A log −series of the variable x is an infinite series of the form

$$\sum_{m_0=1}^{+\infty}\left(\sum_{i=0}^{n} a_i\left(m_0,\ldots,m_n\right)\varepsilon^i\right)x^{\sum\limits_{i=0}^{n} m_i\varepsilon^i}. \tag{3.4}$$

Here, the multidual number $\sum\limits_{i=0}^{n} a_i\left(m_0,\ldots,m_n\right)\varepsilon^i$ represents the coefficient of the term $x^{\sum\limits_{i=0}^{n} m_i\varepsilon^i}$, where a_i is a sequence of the natural numbers $\left(m_0,\ldots,m_n\right)$. Further, the log −series (3.4) can be also written as

$$x^{\sum\limits_{i=1}^{n} m_i\varepsilon^i}\sum_{m_0=1}^{+\infty}\left(\sum_{i=0}^{n} a_i\left(m_0,\ldots,m_n\right)\varepsilon^i\right)x^{m_0}, \tag{3.5}$$

So, it converges if and only the below real power series converge simultaneously

$$\sum_{m_0=1}^{+\infty} a_i\left(m_0,\ldots,m_n\right)x^{m_0},\ i=0,\ldots,n. \tag{3.6}$$

Moreover, it is well known, see [16], that the term $x^{\sum\limits_{i=0}^{n} m_i\varepsilon^i}$ is only defined for $x\geq 0$ such that

$$x^{\sum\limits_{i=1}^{n} m_i\varepsilon^i}=\begin{cases}0 & \text{if } x=0,\\ 1+\sum\limits_{i=1}^{n}\sum\limits_{j=1}^{i} P_{ij}\left(m_1\log x,\ldots,m_i\log x\right)\varepsilon^i & \text{if } x>0.\end{cases} \tag{3.7}$$

Denote now by R_i, $i=0,\ldots,n$, the radius of convergence of the real power series (3.6), respectively. Obviously, the log −series (3.4) converges for every $x\in[0,\ R[$, where

$$R=\min_{i=1,\ldots,n} R_i.$$

On the other hand, by utilizing (3.5) and (3.7), we get

$$\begin{aligned}&x^{\sum\limits_{i=1}^{n} m_i\varepsilon^i}\sum_{m_0=1}^{+\infty}\left(\sum_{i=0}^{n} a_i\left(m_0,\ldots,m_n\right)\varepsilon^i\right)x^{m_0}\\ &=\left(1+\sum_{i=1}^{n}\sum_{j=1}^{i} P_{ij}\left(m_1\log x,\ldots,m_i\log x\right)\varepsilon^i\right)\sum_{m_0=1}^{+\infty}\left(\sum_{i=0}^{n} a_i\left(m_0,\ldots,m_n\right)\varepsilon^i\right)x^{m_0}.\end{aligned}$$

Hence, the fact that P_{ij} are real polynomials allows us to obtain

$$\lim_{x\longrightarrow 0}\sum_{m_0=1}^{+\infty}\left(\sum_{i=0}^{n} a_i\left(m_0,\ldots,m_n\right)\varepsilon^i\right)x^{\sum\limits_{i=0}^{n} m_i\varepsilon^i}=0.$$

We conclude that if the log −series (3.4) converges then its limit is a continuous function at 0. If the log −series converges, its sum is said to be a log −function. We focus ourselves to the study of some elementary log −functions and their applications.

3.1. log −Exponential Function

Given $m=\sum\limits_{i=0}^{n}m_i\varepsilon^i\in\mathbb{Z}_n^+(\varepsilon)$. Let us consider the log −series given by

$$1+\sum_{m_0=1}^{+\infty}\frac{x^{\sum\limits_{i=0}^{n}m_i\varepsilon^i}}{\left(\sum\limits_{i=0}^{n}m_i\varepsilon^i\right)!}. \tag{3.8}$$

It is easy to verify that the log −series converges for every $x\in[0,\ +\infty[$. The sum of the series denoted by $\exp_{m_1,\ldots,m_n}$ is called the log −exponential function. The following result suggest us an explicit expression of the log −exponential function.

Theorem 3.4. *The* log −*exponential function* $\exp_{m_1,\ldots,m_n}$ *can be written for every* $x\in[0,+\infty[$

$$\exp_{m_1,\ldots,m_n}(x)=e^x\left(1+\sum_{i=1}^{n}\sum_{j=1}^{i}\int\limits_0^x e^{-s}P_{ij}\left(m_1\log s,\ldots,m_i\log s\right)ds\varepsilon^i\right) \tag{3.9}$$

Proof. We get by differentiating the log −series (3.8) and using the properties of the multidual factorial

$$\frac{d\exp_{m_1,\ldots,m_n}(x)}{dx}=\exp_{m_1,\ldots,m_n}(x)+x^{\sum\limits_{i=1}^{n}m_i\varepsilon^i}-1.$$

This leads to

$$\frac{d\exp_{m_1,\ldots,m_n}(x)}{dx}=\exp_{m_1,\ldots,m_n}(x)+\sum_{i=1}^{n}\sum_{j=1}^{i}P_{ij}\left(m_1\log x,\ldots,m_i\log x\right)\varepsilon^i.$$

Consequently, the result can be easily achieved making use of the standard argument regarding ordinary differential equations. ∎

One deduces that the formula (3.9) can be also written as

$$\exp_{m_1,\ldots,m_n}(x)=\int\limits_0^x s^{\sum\limits_{i=1}^{n}m_i\varepsilon^i}e^{x-s}ds,\quad \forall x\in[0,\ +\infty[. \tag{3.10}$$

An interesting application of the log −exponential function will be subject of the following result.

Theorem 3.5. *For every* $x\in[0,+\infty[$, *the following formulas hold*

$$\int\limits_0^x e^{x-s}\log s\ ds=(e^x-1)\log x-\sum_{m_0=1}^{+\infty}\frac{H_{m_0-1,1}}{m_0!}x^{m_0}$$

and for $i = 2, \ldots, n$,

$$\sum_{j=1}^{i} \int_{0}^{x} e^{x-s} P_{ij}\left(m_1 \log s, \ldots, m_i \log s\right) ds$$
$$= \sum_{m_0=1}^{+\infty} \sum_{j=1}^{i} P_{ij}\left(m_1 \log x - Y_1\left(m_0\right), \ldots, m_i \log x - Y_i\left(m_0\right)\right) \frac{x^{m_0}}{m_0!},$$

where $Y_i\left(m_0\right)$, $i = 1, \ldots, n$, *are given by (3.3).*

Proof. The function $\exp_{m_1,\ldots,m_n}(x)$ can be written making use of (3.2)

$$\exp_{m_1,\ldots,m_n}(x) = 1 + \sum_{m_0=1}^{+\infty} \frac{x^{\sum\limits_{i=0}^{n} m_i \varepsilon^i}}{m_0!\left(1 + \sum\limits_{i=1}^{n} \sum\limits_{j=1}^{i} P_{ij}\left(Y_1\left(m_0\right), \ldots, Y_i\left(m_0\right)\right) \varepsilon^i\right)}.$$

Since $1 + \sum\limits_{i=1}^{n} \sum\limits_{j=1}^{i} P_{ij}\left(Y_1\left(m_0\right), \ldots, Y_i\left(m_0\right)\right) \varepsilon^i = e^{\sum\limits_{i=1}^{n} Y_i(m_0)\varepsilon^i}$, we can infer

$$\exp_{m_1,\ldots,m_n}(x) = 1 + \sum_{m_0=1}^{+\infty} x^{\sum\limits_{i=1}^{n} m_i \varepsilon^i} e^{\sum\limits_{i=1}^{n} -Y_i(m_0)\varepsilon^i} \frac{x^{m_0}}{m_0!}$$
$$= 1 + \sum_{m_0=1}^{+\infty} e^{\sum\limits_{i=1}^{n} (m_i \log x - Y_i(m_0))\varepsilon^i} \frac{x^{m_0}}{m_0!}.$$

Thus, it follows that

$$\exp_{m_1,\ldots,m_n}(x) = e^x + \sum_{m_0=1}^{+\infty} \left(\sum_{i=1}^{n} \sum_{j=1}^{i} P_{ij}\left(m_1 \log x - Y_1\left(m_0\right), \ldots, m_i \log x - Y_i\left(m_0\right)\right) \varepsilon^i\right) \frac{x^{m_0}}{m_0!}. \tag{3.11}$$

Consequently, the proof follows by by combining (3.10) and (3.11). ■

Example 3.6. The case $n = 2$. Here, we have

$$\exp_{m_1,,m_2}(x) = 1 + x^{m_1\varepsilon + m_2\varepsilon^2} \sum_{m_0=1}^{+\infty} \frac{x^{m_0}}{\left(m_0 + m_1\varepsilon + m_2\varepsilon^2\right)!}. \tag{3.12}$$

Moreover, from Proposition 3.2, the multidual factorial can be evaluated for $n = 2$ as follows.

$$\left(m_0 + m_1\varepsilon + m_2\varepsilon^2\right)! = m_0! e^{m_1 \sum\limits_{r=1}^{m_0} \frac{1}{r}\varepsilon + \left(m_2 \sum\limits_{r=1}^{m_0} \frac{1}{r} - \frac{m_1^2}{2} \sum\limits_{r=1}^{m_0} \frac{1}{r^2}\right)\varepsilon^2}$$
$$= m_0! \left[1 + m_1 H_{m_0-1,1}\varepsilon + \left(\frac{m_1^2}{2}\left(H_{m_0-1,1}^2 - H_{m_0-1,2}\right) + m_2 H_{m_0-1,1}\right)\varepsilon^2\right].$$

Hence, we can infer

$$\frac{x^{m_1\varepsilon+m_2\varepsilon^2}}{(m_0+m_1\varepsilon+m_2\varepsilon^2)!}=\frac{1}{m_0!}\left(1+m_1\log x\varepsilon+\left(\frac{m_1^2}{2}\log x+m_2\log x\right)\varepsilon^2\right)\times$$
$$\left[1-m_1H_{m_0-1,1}\varepsilon+\left(\frac{m_1^2}{2}\left(H_{m_0-1,1}^2+H_{m_0-1,2}\right)-m_2H_{m_0-1,1}\right)\varepsilon^2\right]$$
$$=\quad\frac{1}{m_0!}\left(1+m_1\left(\log x-H_{m_0-1,1}\right)\varepsilon\right)+$$
$$\frac{1}{m_0!}\left(\frac{m_1^2}{2}\left(\log x-2\log xH_{m_0-1,1}+H_{m_0-1,1}^2+H_{m_0-1,2}\right)+\right.$$
$$\left.m_2\left(\log x-H_{m_0-1,1}\right)\right)\varepsilon^2.$$

This yields using (3.9) and (3.12)

$$e^x\left(1+\int_0^x e^{-s}m_1\log s ds\varepsilon+\int_0^x e^{-s}\left(\frac{m_1^2}{2}\log^2 s+m_2\log s\right)ds\varepsilon^2\right)$$
$$=1+\sum_{m_0=1}^{+\infty}\left(1+m_1\left(\log x+H_{m_0-1,1}\right)\right)\varepsilon\frac{x^{m_0}}{m_0!}+$$
$$\sum_{m_0=1}^{+\infty}\left(\frac{m_1^2}{2}\left(\log x-2\log xH_{m_0-1,1}+H_{m_0-1,1}^2+H_{m_0-1,2}\right)+\right.$$
$$\left.m_2\left(\log x-H_{m_0-1,1}\right)\varepsilon^2\right)\frac{x^{m_0}}{m_0!}.$$

Consequently, the following formulas hold

$$\int_0^x e^{x-s}\log s\,ds=(e^x-1)\log x-\sum_{m_0=1}^{+\infty}H_{m_0-1,1}\frac{x^{m_0}}{m_0!}.$$
$$\int_0^x e^{x-s}\log^2 s\,ds=(e^x-1)\log x-2\log x\sum_{m_0=1}^{+\infty}\frac{H_{m_0-1,1}}{m_0!}x^{m_0}+$$
$$\sum_{m_0=1}^{+\infty}\frac{H_{m_0-1,1}^2+H_{m_0-1,2}}{m_0!}x^{m_0}.$$

3.2. log −Trigonometric Functions

Let $m=\sum_{i=0}^{n}m_i\varepsilon^i\in\mathbb{Z}_n^+(\varepsilon)$. Consider the multidual log −series given respectively by

$$\left\{\begin{array}{c}1+\sum_{m_0=1}^{+\infty}(-1)^{m_0}\frac{x^{2m_0+\sum_{i=1}^{n}m_i\varepsilon^i}}{\left(2m_0+\sum_{i=1}^{n}m_i\varepsilon^i\right)!}\\ \text{and}\\ \sum_{m_0=0}^{+\infty}(-1)^{m_0}\frac{x^{2m_0+1+\sum_{i=1}^{n}m_i\varepsilon^i}}{\left(2m_0+1+\sum_{i=1}^{n}m_i\varepsilon^i\right)!}\end{array}\right..$$

The both series converge for every $x \in [0, +\infty[$. Their sums are denoted respectively by $\cos_{m_1,...,m_n}$ and $\sin_{m_1,...,m_n}$ called $\log -$cosinus and $\log -$sinus functions. The following result suggests some properties.

Theorem 3.7. *The* $\log -$*trigonometric functions* $\cos_{m_1,...,m_n}$ *and* $\sin_{m_1,...,m_n}$ *verify the differential equations*

$$\frac{d^2 \cos_{m_1,...,m_n}(x)}{dx^2} + \cos_{m_1,...,m_n}(x) = -\sum_{i=1}^{n}\sum_{j=1}^{i} P_{ij}(m_1 \log x, ..., m_i \log x)\,\varepsilon^i. \tag{3.13}$$

$$\frac{d \cos_{m_1,...,m_n}(x)}{dx} = -\sin_{m_1,...,m_n}(x). \tag{3.14}$$

$$\frac{d \sin_{m_1,...,m_n}(x)}{dx} = \cos_{m_1,...,m_n}(x) + \sum_{i=1}^{n}\sum_{j=1}^{i} P_{ij}(m_1 \log x, ..., m_i \log x)\,\varepsilon^i.$$

Proof. It is straightforward to find by computing the first and second derivative of the function $\cos_{m_1,...,m_n}$ and using the properties of the multidual factorial

$$\frac{d \cos_{m_1,...,m_n}(x)}{dx} = -\sin_{m_1,...,m_n}(x) \text{ and}$$
$$\frac{d^2 \cos_{m_1,...,m_n}(x)}{dx^2} = -\cos_{m_1,...,m_n}(x) - x^{\sum\limits_{i=1}^{n} m_i\varepsilon^i} + 1.$$

These also give

$$\frac{d \sin_{m_1,...,m_n}(x)}{dx} = -\frac{d^2 \cos_{m_1,...,m_n}(x)}{dx^2} = \cos_{m_1,...,m_n}(x) + x^{\sum\limits_{i=1}^{n} m_i\varepsilon^i} - 1.$$

Thus, the desired result follows. ∎

The subject of the following Theorem is to determinate explicit expression of the functions $\cos_{m_1,...,m_n}$ and $\sin_{m_1,...,m_n}(x)$.

Theorem 3.8. *The* $\log -$*trigonometric functions* $\cos_{m_1,...,m_n}$ *and* $\sin_{m_1,...,m_n}(x)$ *verify the below formulas*

$$\cos_{m_1,...,m_n}(x) = \cos x + \sum_{i=1}^{n}\sum_{j=1}^{i}\left(\cos x \int_0^x \sin s P_{ij}(m_1 \log s, ..., m_i \log s)\,ds \right.$$
$$\left. - \sin x \int_0^x \cos s P_{ij}(m_1 \log s, ..., m_i \log s)\,ds\right)\varepsilon^i.$$

$$\sin_{m_1,...,m_n}(x) = \sin x + \sum_{i=1}^{n}\sum_{j=1}^{i}\left(\sin x \int_0^x \sin s P_{ij}(m_1 \log s, ..., m_i \log s)\, ds \right.$$
$$\left. + \cos x \int_0^x \cos s P_{ij}(m_1 \log s, ..., m_i \log s)\, ds\right)\varepsilon^i.$$

Proof. Setting $\cos_{m_1,...,m_n}(x) = y$ in the ordinary differential equation (3.13). It is well known that the general solution of the homogeneous corresponding equation is

$$y = c_1 \cos x + c_2 \sin x.$$

Using the method of variation of parameters, the particular solution is formed by replacing in the general solution the parameters c_1 and c_2 by unknown functions $c_1(x)$ and $c_2(x)$. So the particular, the solution can be written as

$$y_p = c_1(x)\cos x + c_2(x)\sin x.$$

It is reasonable to impose, after computing the derivative of the particular solution

$$c_1'(x)\cos x + c_2'(x)\sin x = 0.$$

Differentiating this equation and substituting the obtained one in the particular solution we get

$$-c_1'(x)\sin x + c_2'(x)\cos x = -\sum_{i=1}^{n}\sum_{j=1}^{i} P_{ij}(m_1 \log x, \ldots, m_i \log x)\,\varepsilon^i.$$

Solving the above system allows as to determinate the solution of the inhomogeneous equation. Indeed, we have

$$\cos_{m_1,...,m_n}(x) = \cos x + \sum_{i=1}^{n}\sum_{j=1}^{i}\left(\cos x \int_0^x \sin s P_{ij}(m_1 \log s, ..., m_i \log s)\, ds \right.$$
$$\left. - \sin x \int_0^x \cos s P_{ij}(m_1 \log s, ..., m_i \log s)\, ds\right)\varepsilon^i + c_1 \cos x + c_2 \sin x.$$

Combining the previous equation with equation (3.14), we also get

$$\sin_{m_1,...,m_n}(x) = \sin x + \sum_{i=1}^{n}\sum_{j=1}^{i}\left(\sin x \int_0^x \sin s P_{ij}(m_1 \log s, ..., m_i \log s)\, ds \right.$$
$$\left. + \cos x \int_0^x \cos s P_{ij}(m_1 \log s, ..., m_i \log s)\, ds\right)\varepsilon^i - c_1 \sin x + c_2 \cos x.$$

By the definition of the function $\cos_{m_1,...,m_n}$ and $\sin_{m_1,...,m_n}$, we have

$$\cos_{m_1,...,m_n}(0) = 1 \quad \text{and} \quad \sin_{m_1,...,m_n}(0) = 0.$$

So, we find $c_1 = c_2 = 0$ which permits us to conclude the proof. ∎

It is straightforward to show that the previous formulas can be also written in compact form

$$\begin{cases} \cos_{m_1,\ldots,m_n}(x) = 1 + \int\limits_0^x s^{\sum\limits_{i=1}^n m_i\varepsilon^i} \sin(s-x)\,ds, \\ \sin_{m_1,\ldots,m_n}(x) = \int\limits_0^x s^{\sum\limits_{i=1}^n m_i\varepsilon^i} \cos(s-x)\,ds. \end{cases} \tag{3.15}$$

We provide in the following an application of the log −trigonometric functions.

Theorem 3.9. *For all $x \in [0,+\infty[$ and $i = 1,\ldots,n$, the following formulas hold*

$$\sum_{j=1}^{i} \int_0^x \sin(s-x)\,P_{ij}(m_1\log s,\ldots,m_i\log s)\,ds$$
$$= \sum_{m_0=1}^{+\infty}\sum_{j=1}^{i}(-1)^{m_0} P_{ij}(m_1\log x - Y_1(m_0),\ldots,m_i\log x - Y_i(m_0))\frac{x^{2m_0}}{(2m_0)!}.$$

and

$$\sum_{j=1}^{i} \int_0^x \cos(s-x)\,P_{ij}(m_1\log s,\ldots,m_i\log s)\,ds$$
$$= \sum_{m_0=0}^{+\infty}\sum_{j=1}^{i}(-1)^{m_0} P_{ij}(m_1\log x - Y_1(m_0),\ldots,m_i\log x - Y_i(m_0))\frac{x^{2m_0+1}}{(2m_0+1)!}.$$

Proof. Using (3.2), the functions $\cos_{m_1,\ldots,m_n}(x)$ and $\sin_{m_1,\ldots,m_n}(x)$ can be written as

$$\cos_{m_1,\ldots,m_n}(x) = 1 + \sum_{m_0=1}^{+\infty}(-1)^{m_0}\frac{x^{2m_0+\sum\limits_{i=1}^n m_i\varepsilon^i}}{(2m_0)!e^{\sum\limits_{i=1}^n Y_i(2m_0)\varepsilon^i}},$$

and

$$\sin_{m_1,\ldots,m_n}(x) = \sum_{m_0=0}^{+\infty}(-1)^{m_0}\frac{x^{2m_0+\sum\limits_{i=1}^n m_i\varepsilon^i}}{(2m_0+1)!e^{\sum\limits_{i=1}^n Y_i(2m_0+1)\varepsilon^i}}.$$

Thus we find

$$\cos_{m_1,\ldots,m_n}(x) = 1 + \sum_{m_0=1}^{+\infty}(-1)^{m_0} e^{\sum\limits_{i=1}^n (m_i\log x - Y_i(2m_0))\varepsilon^i}\frac{x^{2m_0}}{(2m_0)!},$$
$$= \cos x + \sum_{m_0=1}^{+\infty}(-1)^{m_0}\left(\sum_{i=1}^{n}\sum_{j=1}^{i} P_{ij}(m_1\log x - Y_1(2m_0)\right.$$
$$\left.,\ldots,m_i\log x - Y_i(2m_0))\,\varepsilon^i\right)\frac{x^{2m_0}}{(2m_0)!}$$

and

$$\begin{aligned}
&\sin_{m_1,\ldots,m_n}(x) \\
&= \sum_{m_0=0}^{+\infty}(-1)^{m_0} e^{\sum\limits_{i=1}^{n}(m_i\log x - Y_i(2m_0+1))\varepsilon^i}\frac{x^{2m_0+1}}{(2m_0+1)!} \\
&= \sin x + \sum_{m_0=0}^{+\infty}(-1)^{m_0}\left(\sum_{i=1}^{n}\sum_{j=1}^{i}P_{ij}\left(m_1\log x - Y_1\left(2m_0+1\right),\ldots,\right.\right. \\
&\qquad \left.m_i\log x - Y_i\left(2m_0+1\right)\right)\varepsilon^i\Big)\frac{x^{2m_0+1}}{(2m_0+1)!}.
\end{aligned}$$

Consequently, the proof follows keeping in mind the formula (3.15). ∎

Example 3.10. The case $n=2$. Here, we have

$$\begin{cases}
\cos_{m_1,m_2}(x) = 1 + \sum\limits_{m_0=1}^{+\infty}(-1)^{m_0}\frac{x^{2m_0+m_1\varepsilon+m_2\varepsilon^2}}{(2m_0+m_1\varepsilon+m_2\varepsilon^2)!}, \\
\text{and} \\
\sin_{m_1,m_2}(x) = \sum\limits_{m_0=0}^{+\infty}(-1)^{m_0}\frac{x^{2m_0+1+m_1\varepsilon+m_2\varepsilon^2}}{(2m_0+1+m_1\varepsilon+m_2\varepsilon^2)!}.
\end{cases}$$

So, one can easily find

$$\begin{aligned}
&\cos_{m_1,m_2}(x) \\
&= 1 + x^{2m_0+m_1\varepsilon+m_2\varepsilon^2}\sum_{m_0=1}^{+\infty}(-1)^{m_0}\frac{x^{2m_0}}{(2m_0)!}\times \\
&\quad \left[1 - m_1H_{2m_0-1,1}\varepsilon + \left(\frac{m_1^2}{2}\left(H_{2m_0-1,1}^2 + H_{2m_0-1,2}\right) - m_2H_{2m_0-1_0,1}\right)\varepsilon^2\right],
\end{aligned}$$

and

$$\begin{aligned}
&\sin_{m_1,m_2}(x) \\
&= x^{2m_0+1+m_1\varepsilon+m_2\varepsilon^2}\sum_{m_0=0}^{+\infty}(-1)^{m_0}\frac{x^{2m_0+1}}{(2m_0+1)!}\times \\
&\quad \left[1 - m_1H_{2m_0,1}\varepsilon + \left(\frac{m_1^2}{2}\left(H_{2m_0,1}^2 + H_{2m_0,2}\right) - m_2H_{2m_0,1}\right)\varepsilon^2\right].
\end{aligned}$$

Hence, taking into account (3.15), we obtain the following formulas

$$\int_0^x \sin(s-x)\log s\,ds = (\cos x - 1)\log x - \sum_{m_0=1}^{+\infty}(-1)^{m_0}\frac{H_{2m_0-1,1}}{(2m_0)!}x^{2m_0},$$

$$\begin{aligned}
\int_0^x \sin(s-x)\log^2 s\,ds &= (\cos x - 1)\log^2 x - 2\log x\sum_{m_0=1}^{+\infty}(-1)^{m_0}\frac{H_{2m_0-1,1}}{(2m_0)!}x^{2m_0} \\
&\quad + \sum_{m_0=1}^{+\infty}(-1)^{m_0}\frac{H_{2m_0-1,1}^2 + H_{2m_0-1,2}}{(2m_0)!}x^{2m_0},
\end{aligned}$$

$$\int_0^x \cos(s-x)\log s ds = \sin x \log x - \sum_{m_0=0}^{+\infty} (-1)^{m_0} \frac{H_{2m_0,1}}{(2m_0+1)!} x^{2m_0+1},$$

$$\int_0^x \cos(s-x)\log^2 s ds = \sin x \log^2 x - 2\log x \sum_{m_0=0}^{+\infty} (-1)^{m_0} \frac{H_{2m_0,1}}{(2m_0+1)!} x^{2m_0+1}$$
$$+ \sum_{m_0=0}^{+\infty} (-1)^{m_0} \frac{H^2_{2m_0,1}(1) + H_{2m_0,2}(1)}{(2m_0+1)!} x^{2m_0+1}.$$

3.3. log −Hyperbolic Functions

Let $m = \sum_{i=0}^{n} m_i \varepsilon^i \in \mathbb{Z}_n^+(\varepsilon)$. Considering the log −series given respectively by

$$\begin{cases} 1 + \sum_{m_0=1}^{+\infty} \frac{x^{2m_0+\sum_{i=0}^{n} m_i\varepsilon^i}}{\left(2m_0+\sum_{i=0}^{n} m_i\varepsilon^i\right)!}, \\ \text{and} \\ \sum_{m_0=0}^{+\infty} \frac{x^{2m_0+1+\sum_{i=0}^{n} m_i\varepsilon^i}}{\left(2m_0+1+\sum_{i=0}^{n} m_i\varepsilon^i\right)!}. \end{cases}$$

These two series converge for every $x \in [0, +\infty[$. Their sums are denoted respectively by $\cosh_{m_1,\ldots,m_n}$ and $\sinh_{m_1,\ldots,m_n}$ called respectively log −hyperbolic cosinus and log-hyperbolic sinus functions.

Theorem 3.11. *The following formulas hold*

$$\frac{d^2\cosh_{m_1,\ldots,m_n}(x)}{dx^2} - \cosh_{m_1,\ldots,m_n}(x) = \sum_{i=1}^{n}\sum_{j=1}^{i} P_{ij}(m_1\log x,\ldots,m_i\log x)\,\varepsilon^i.$$
$$\frac{d\cosh_{m_1,\ldots,m_n}(x)}{dx} = \sinh_{m_1,\ldots,m_n}(x). \tag{3.16}$$
$$\frac{d\sinh_{m_1,\ldots,m_n}(x)}{dx} = \cosh_{m_1,\ldots,m_n}(x) + \sum_{i=1}^{n}\sum_{j=1}^{i} P_{ij}(m_1\log x,\ldots,m_i\log x)\,\varepsilon^i.$$

Proof. Making use of the properties of the multidual factorial, we can easily obtain, by differentiating the function $\cosh_{m_1,\ldots,m_n}(x)$

$$\frac{d\cosh_{m_1,\ldots,m_n}(x)}{dx} = \sinh_{m_1,\ldots,m_n}(x)$$

and

$$\frac{d^2\cosh_{m_1,\ldots,m_n}(x)}{dx^2} = \cosh_{m_1,\ldots,m_n}(x) + x^{\sum_{i=1}^{n} m_i\varepsilon^i} - 1.$$

Thus, we get

$$\frac{d\sinh_{m_1,\ldots,m_n}(x)}{dx} = \frac{d^2\cosh_{m_1,\ldots,m_n}(x)}{dx^2} = \cosh_{m_1,\ldots,m_n}(x) + x^{\sum_{i=1}^{n} m_i\varepsilon^i} - 1,$$

which achieves the proof. ■

The goal of the following result is to determinate the explicit expression of the functions $\cosh_{m_1,\ldots,m_n}$ and $\sinh_{m_1,\ldots,m_n}$.

Theorem 3.12. *The* log −*hyperbolic functions* $\cosh_{m_1,\ldots,m_n}$ *and* $\sinh_{m_1,\ldots,m_n}$ *are given explicitely by:*

$$\cosh_{m_1,\ldots,m_n}(x) = \cosh x + \sum_{i=1}^{n}\sum_{j=1}^{i}\left(-\cosh x\int_0^x \sinh s P_{ij}(m_1\log s,\ldots,m_i\log s)\,ds + \sinh x\int_0^x \cosh s P_{ij}(m_1\log s,\ldots,m_i\log s)\,ds\right)\varepsilon^i. \tag{3.17}$$

$$\sinh_{m_1,\ldots,m_n}(x) = \sinh x + \sum_{i=1}^{n}\sum_{j=1}^{i}\left(-\sinh x\int_0^x \sinh s P_{ij}(m_1\log s,\ldots,m_i\log s)\,ds + \cosh x\int_0^x \cosh s P_{ij}(m_1\log s,\ldots,m_i\log s)\,ds\right)\varepsilon^i.$$

Proof. Let us denote by y the solution of the ODE (3.17). The general solution of the homogeneous corresponding equation is given by

$$y = c_1\cosh x + c_2\sinh x.$$

To determinate the particular solution of the inhomogeneous equation we use the method variation of parameters. Indeed, the particular solution is assumed to be of the form

$$y_p = c_1(x)\cosh x + c_2(x)\sinh x,$$

where $c_1(x)$ and $c_2(x)$ are unknown functions. We choose the following

$$\begin{cases} c_1'(x)\cosh x + c_2'(x)\sinh x = 0, \\ c_1'(x)\sinh x + c_2'(x)\cosh x = \sum\limits_{i=1}^{n}\sum\limits_{j=1}^{i} P_{ij}(m_1\log x,\ldots,m_i\log x)\,\varepsilon^i. \end{cases}$$

So, by solving the above system we deduce that the solution of the inhomogeneous equation can be represented by the function

$$\cosh_{m_1,\ldots,m_n}(x) = \cosh x + \sum_{i=1}^{n}\sum_{j=1}^{i}\left(-\cosh x\int_0^x \sinh s P_{ij}(m_1\log s,\ldots,m_i\log s)\,ds + \sinh x\int_0^x \cosh s P_{ij}(m_1\log s,\ldots,m_i\log s)\,ds\right)\varepsilon^i + c_1\cosh x + c_2\sinh x.$$

Differentiating this equation, we get, keeping in mind (3.16)

$$\sinh_{m_1,\dots,m_n}(x) = \sinh x + \sum_{i=1}^{n}\sum_{j=1}^{i}\Bigg(-\sinh x\int_0^x \sinh s P_{ij}(m_1\log s,...,m_i\log s)\,ds$$
$$+\cosh x\int_0^x \cosh s P_{ij}(m_1\log s,...,m_i\log s)\,ds\Bigg)\varepsilon^i + c_1\sinh x + c_2\cosh x.$$

It is clear that

$$\cosh_{m_1,\dots,m_n}(0) = 1 \quad \text{and} \quad \sinh_{m_1,\dots,m_n}(0) = 0.$$

It follows that $c_1 = c_2 = 0$. Consequently, the Theorem is proved. ■

It is straightforward to show that the previous formulas can be written in compact form

$$\begin{cases} \cosh_{m_1,\dots,m_n}(x) = 1 + \int\limits_0^x s^{\sum\limits_{i=1}^{n} m_i\varepsilon^i}\sinh(x-s)\,ds, \\ \sinh_{m_1,\dots,m_n}(x) = \int\limits_0^x s^{\sum\limits_{i=1}^{n} m_i\varepsilon^i}\cosh(x-s)\,ds. \end{cases}$$

An application of the log −hyperbolic functions in the field of special functions will be given in the following assertion.

Theorem 3.13. *For all $x\in[0,+\infty[$ and $i=1,\dots,n$, the following formulas hold.*

$$\sum_{j=1}^{i}\int_0^x \sinh(x-s)\,P_{ij}(m_1\log s,...,m_i\log s)\,ds$$
$$= \sum_{m_0=1}^{+\infty}\sum_{j=1}^{i}P_{ij}(m_1\log x - Y_1(m_0),...,m_i\log x - Y_i(m_0))\frac{x^{2m_0}}{(2m_0)!},$$

$$\sum_{j=1}^{i}\int_0^x \cosh(x-s)\,P_{ij}(m_1\log s,...,m_i\log s)\,ds$$
$$= \sum_{m_0=0}^{+\infty}\sum_{j=1}^{i}P_{ij}(m_1\log x - Y_1(m_0),...,m_i\log x - Y_i(m_0))\frac{x^{2m_0+1}}{(2m_0+1)!}.$$

Proof. The functions $\cosh_{m_1,\dots,m_n}$ and $\sinh_{m_1,\dots,m_n}(x)$ can be written making use (3.2)

$$\cosh_{m_1,\dots,m_n}(x) = 1 + \sum_{m_0=1}^{+\infty}\frac{x^{2m_0+\sum\limits_{i=1}^{n} m_i\varepsilon^i}}{(2m_0)!\,e^{\sum\limits_{i=1}^{n} Y_i(2m_0)\varepsilon^i}}$$

and

$$\sinh_{m_1,\dots,m_n}(x) = \sum_{m_0=0}^{+\infty}\frac{x^{2m_0+\sum\limits_{i=1}^{n} m_i\varepsilon^i}}{(2m_0+1)!\,e^{\sum\limits_{i=1}^{n} Y_i(2m_0+1)\varepsilon^i}}.$$

Thus we find

$$\cosh_{m_1,\ldots,m_n}(x) = 1 + \sum_{m_0=1}^{+\infty} e^{\sum\limits_{i=1}^{n}(m_i \log x - Y_i(2m_0))\varepsilon^i} \frac{x^{2m_0}}{(2m_0)!},$$

$$= \cosh x + \sum_{m_0=1}^{+\infty} \left(\sum_{i=1}^{n} \sum_{j=1}^{i} P_{ij} \left(m_1 \log x - Y_1 \left(2m_0\right)\right.\right.$$

$$\left.\left., \ldots, m_i \log x - Y_i \left(2m_0\right)\right) \varepsilon^i \right) \frac{x^{2m_0}}{(2m_0)!},$$

and

$$\sinh_{m_1,\ldots,m_n}(x) = \sum_{m_0=0}^{+\infty} e^{\sum\limits_{i=1}^{n}(m_i \log x - Y_i(2m_0+1))\varepsilon^i} \frac{x^{2m_0+1}}{(2m_0+1)!}$$

$$= \sin x + \sum_{m_0=0}^{+\infty} \left(\sum_{i=1}^{n} \sum_{j=1}^{i} P_{ij} \left(m_1 \log x - Y_1 \left(2m_0+1\right), \ldots,\right.\right.$$

$$\left.\left. m_i \log x - Y_i \left(2m_0+1\right)\right) \varepsilon^i \right) \frac{x^{2m_0+1}}{(2m_0+1)!}.$$

These yield exploiting

$$1 + \int_0^x \left(1 + \sum_{i=1}^{n} \sum_{j=1}^{i} P_{ij} \left(m_1 \log x, \ldots, m_i \log x\right) \varepsilon^i \right) \sinh\left(x - s\right) ds$$

$$= \sum_{m_0=1}^{+\infty} \left(\sum_{i=1}^{n} \sum_{j=1}^{i} P_{ij} \left(m_1 \log x - Y_1 \left(2m_0\right), \ldots, m_i \log x - Y_i \left(2m_0\right)\right) \varepsilon^i \right) \frac{x^{2m_0}}{(2m_0)!},$$

and

$$\int_0^x \left(1 + \sum_{i=1}^{n} \sum_{j=1}^{i} P_{ij} \left(m_1 \log x, \ldots, m_i \log x\right) \varepsilon^i \right) \cosh\left(x - s\right) ds$$

$$= \sum_{m_0=0}^{+\infty} \left(\sum_{i=1}^{n} \sum_{j=1}^{i} P_{ij} \left(m_1 \log x - Y_1 \left(2m_0+1\right), \ldots,\right.\right.$$

$$\left.\left. m_i \log x - Y_i \left(2m_0+1\right)\right) \varepsilon^i \right) \frac{x^{2m_0+1}}{(2m_0+1)!},$$

which completes the proof. ∎

Example 3.14. The case $n = 2$. Here, we have

$$\begin{cases} \cosh_{m_1,m_2}(x) = 1 + \sum\limits_{m_0=1}^{+\infty} \frac{x^{2m_0+m_1\varepsilon+m_2\varepsilon^2}}{(2m_0+m_1\varepsilon+m_2\varepsilon^2)!}, \\ \sinh_{m_1,m_2}(x) = \sum\limits_{m_0=0}^{+\infty} \frac{x^{2m_0+1+m_1\varepsilon+m_2\varepsilon^2}}{(2m_0+1+m_1\varepsilon+m_2\varepsilon^2)!}. \end{cases}$$

Then, one finds

$$\cosh_{m_1,m_2}(x) = 1 + x^{2m_0+m_1\varepsilon+m_2\varepsilon^2}\sum_{m_0=1}^{+\infty}\frac{x^{2m_0}}{(2m_0)!}\times$$
$$\left[1 - m_1 H_{2m_0-1,1}\varepsilon + \left(\frac{m_1^2}{2}\left(H_{2m_0-1,1}^2 + H_{2m_0-1,2}\right) - m_2 H_{2m_0-1_0,1}\right)\varepsilon^2\right]$$

and

$$\sinh_{m_1,m_2}(x) = x^{2m_0+1+m_1\varepsilon+m_2\varepsilon^2}\sum_{m_0=0}^{+\infty}\frac{x^{2m_0+1}}{(2m_0+1)!}\times$$
$$\left[1 - m_1 H_{2m_0,1}\varepsilon + \left(\frac{m_1^2}{2}\left(H_{2m_0,1}^2 + H_{2m_0,2}\right) - m_2 H_{2m_0,1}\right)\varepsilon^2\right].$$

Hence, keeping in mind Theorem 3.13, we obtain the following formulas

$$\int_0^x \sinh(s-x)\log s ds = (\cosh x - 1)\log x - \sum_{m_0=1}^{+\infty}\frac{H_{2m_0-1,1}}{(2m_0)!}x^{2m_0},$$
$$\int_0^x \sinh(s-x)\log^2 s ds = (\cosh x - 1)\log^2 x - 2\log x\sum_{m_0=1}^{+\infty}\frac{H_{2m_0-1,1}}{(2m_0)!}x^{2m_0},$$
$$+\sum_{m_0=1}^{+\infty}\frac{H_{2m_0-1,1}^2 + H_{2m_0-1,2}}{(2m_0)!}x^{2m_0},$$
$$\int_0^x \cosh(s-x)\log s ds = \sinh x\log x - \sum_{m_0=0}^{+\infty}\frac{H_{2m_0,1}}{(2m_0+1)!}x^{2m_0+1},$$
$$\int_0^x \cosh(s-x)\log^2 s ds = \sinh x\log^2 x - 2\log x\sum_{m_0=0}^{+\infty}\frac{H_{2m_0,1}}{(2m_0+1)!}x^{2m_0+1},$$
$$+\sum_{m_0=0}^{+\infty}\frac{H_{2m_0,1}^2 + H_{2m_0,2}}{(2m_0+1)!}x^{2m_0+1}.$$

4. Conclusion

In this paper, we introduced the so-called $\log -$series. The idea was to consider a real power series and replace the natural powers with multidual integers and the coefficient by a multidual sequence. The sum of a $\log -$serie is said to be $\log -$function. We have studied some elementary $\log -$function, namely the $\log -$exponential function, the $\log -$trigonometric functions and the $\log -$hyperbolic functions, as generalization of the real elementary functions. It has been shown that we can use $\log -$functions and tools of multidual analysis to obtain expansion of some special functions in series involving $n-$th power of the Logarithmic function and Harmonic numbers.

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