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Three Inequalities for Quadratic-Phase Fourier Transform

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Abstract In this work, we introduce the one-dimensional quadratic-phase Fourier transform. The relation between the one-dimensional quadratic-phase Fourier transform and the one-dimensional Fourier transform is discussed in detail. We finally propose several versions of the inequalities related to one-dimensional quadratic-phase Fourier transform.

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1. Introduction

The quadratic-phase Fourier transform [6, 7] is a generalization of many transformations such as the Fourier transform [4, 9], fractional Fourier transform [1–3, 5, 8, 10, 11], linear canonical transform, and special affine Fourier transform. On the other hand, the uncertainty principle plays an important role in signal processing. It describes a function and its Fourier transform, which cannot both be simultaneously sharply localized. Uncertainty principles have implications in two main areas: quantum physics and signal analysis. In quantum physics, they tell us that a particle and position cannot both be measured with infinite precision. In signal analysis, they tell us that if we observe a signal only for a finite period of time, we will lose information about the frequencies of the signal. In the recent past, many works have been devoted to establishing some uncertainty principles in different settings of various transforms. In this work, we establish three inequalities related to the quadratic phase Fourier transform.

2. Preliminaries and Basic Facts

In this part, we recall some notations, definitions, and preliminary facts which are needed throughout this work.

Definition 2.1. The one-dimensional quadratic-phase Fourier transform (1DQPFT) of a function $f \in L^2(\mathbb{R})$ is given by the integral

$$\mathcal{Q}_K\{f\}(\omega) = \int_{\mathbb{R}} f(x)K(x,\omega)\, dx,$$

where

$$K(x,\omega) = e^{-i(ax^2+bx\omega+c\omega^2+dx+e\omega)}.$$

Without loss of generality throughout this study, we always assume that $b > 0$. Let $L^p(\mathbb{R})$ denotes the Banach space of integrable functions such that

$$\|f\|_{L^p(\mathbb{R})} = \left(\int_{\mathbb{R}} |f(x)|^p\, dx\right)^{1/p} < \infty, \quad \text{for } p \in [1,\infty)$$

and

$$\|f\|_{L^\infty(\mathbb{R})} = \operatorname*{ess\,sup}_{x\in\mathbb{R}} |f(x)| < \infty, \quad \text{for } p = \infty.$$

It is straightforward to check that the Parseval's formula for the 1DQPFT related to complex functions $f, g \in L^2(\mathbb{R})$ is expressed as

$$(f,g) = \int_{\mathbb{R}} f(x)\overline{g(x)}\, dx = \int_{\mathbb{R}} \mathcal{Q}_K\{f\}(\omega)\overline{\mathcal{Q}_K\{g\}(\omega)}\, d\omega.$$

In particular,

$$\|f\|^2_{L^2(\mathbb{R})} = \|\mathcal{Q}_K\{f\}\|^2_{L^2(\mathbb{R})}.$$

3. Relationship between the Fourier transform and the quadratic-phase quadratic transform

Below, we describe relationship between the Fourier transform and the quadratic-phase transform as follow:

$$\begin{aligned}
\mathcal{Q}_K\{f\}(\omega) &= \int_{\mathbb{R}} f(x)K(x,\omega)\, dx \\
&= \int_{\mathbb{R}} f(x)e^{-i(ax^2+bx\omega+c\omega^2+dx+e\omega)}\, dx \\
&= e^{-i(c\omega^2+e\omega)} \int_{\mathbb{R}} f(x)e^{-i(ax^2+dx)}e^{-ibx\omega}\, dx \\
&= e^{-i(c\omega^2+e\omega)} \int_{\mathbb{R}} g(x)e^{-ibx\omega}\, dx \\
&= e^{-i(c\omega^2+e\omega)}\mathcal{F}\{g\}(b\omega).
\end{aligned}$$

Here

$$g(x) = f(x)e^{-i(ax^2+dx)}. \tag{3.1}$$

Therefore

$$\mathcal{Q}_K\{f\}(\omega)e^{i(c\omega^2+e\omega)} = \mathcal{F}\{g\}(b\omega), \tag{3.2}$$

where $\mathcal{F}\{g\}(\omega)$ is the Fourier transform for a complex function $g \in L^2(\mathbb{R})$ in [4, 9] defined by

$$\mathcal{F}\{g\}(\omega) = \hat{g}(\omega) = \int_{\mathbb{R}} g(x) e^{-ix\omega}\, dx.$$

4. Uncertainty principles for Quadratic-Phase Fourier Transform

In this section, we investigate two uncertainty principles related to the quadratic-phase Fourier transform. The principles can be considered as an extension of uncertainty principles for the Fourier transform.

4.1. Heisenberg's inequality

We obtain the Heisenberg's inequality related to the quadratic-phase Fourier transform in the following result.

Theorem 4.1. *Let $f \in L^2(\mathbb{R})$ be a complex function. If $\mathcal{Q}_K\{f\}(\omega) \in L^2(\mathbb{R})$, we have*

$$\left(\int_{\mathbb{R}} x^p \left|f\left(x\right)\right|^p dx\right)\left(\int_{\mathbb{R}} \omega^p \left|\mathcal{Q}_K\left\{f\right\}\left(\omega\right)\right|^p d\omega\right) \geq \frac{b^{-(p+1)}}{2^p}\left(\int_{\mathbb{R}} \left|f\left(x\right)\right|^2 dx\right)^p.$$

For $1 \leq p \leq 2$.

Proof. It directly follows from the generalization of the Heisenberg's inequality for the Fourier transform that

$$\left(\int_{\mathbb{R}} x^p \left|f\left(x\right)\right|^p dx\right)\left(\int_{\mathbb{R}} \omega^p \left|\mathcal{F}\left\{f\right\}\left(\omega\right)\right|^p d\omega\right) \geq \left(\frac{1}{2}\right)^p\left(\int_{\mathbb{R}} \left|f\left(x\right)\right|^2 dx\right)^p$$

Since $g(t)$ defined by (3.1) belongs to $L^2(\mathbb{R})$, then by replacing $f(t)$ by $g(t)$ into the above identity, we obtain

$$\left(\int_{\mathbb{R}} x^p \left|g\left(x\right)\right|^p dx\right)\left(\int_{\mathbb{R}} \omega^p \left|\mathcal{F}\left\{g\right\}\left(\omega\right)\right|^p d\omega\right) \geq \left(\frac{1}{2}\right)^p\left(\int_{\mathbb{R}} \left|g\left(x\right)\right|^2 dx\right)^p \tag{4.1}$$

Substituting ω with ωb, we see from relations (4.1) that

$$\left(\int_{\mathbb{R}} x^p \left|g\left(x\right)\right|^p dx\right)\left(\int_{\mathbb{R}} (\omega b)^p \left|\mathcal{F}\left\{g\right\}\left(\omega b\right)\right|^p d(\omega b)\right) \geq \left(\frac{1}{2}\right)^p\left(\int_{\mathbb{R}} \left|g\left(x\right)\right|^2 dx\right)^p.$$

From equation (3.2) we get

$$\left(\int_{\mathbb{R}} x^p \left|f\left(x\right)e^{-i\left(ax^2+dx\right)}\right|^p dx\right)\left(\int_{\mathbb{R}} \omega^p b^{p+1} \left|\mathcal{Q}_K\left\{f\right\}\left(\omega\right)e^{i\left(c\omega^2+e\omega\right)}\right|^p d\omega\right)$$
$$\geq \left(\frac{1}{2}\right)^p\left(\int_{\mathbb{R}} \left|f\left(x\right)e^{-i\left(ax^2+dx\right)}\right|^2 dx\right)^p.$$

The above equation is equal to

$$\left(\int_{\mathbb{R}} x^p \left|f\left(x\right)\right|^p dx\right)\left(\int_{\mathbb{R}} \omega^p \left|\mathcal{Q}_K\left\{f\right\}\left(\omega\right)\right|^p d\omega\right) \geq \frac{b^{-(p+1)}}{2^p}\left(\int_{\mathbb{R}} \left|f\left(x\right)\right|^2 dx\right)^p,$$

and the proof is complete. ∎

4.2. Hausdorff-Young Inequality

In this part, we establish Housdorf-Young inequality for the quadratic- phase Fourier Transform.

Theorem 4.2 (Hausdorff-Young inequality). *For any* $1 \leq r \leq 2$ *such that* $\frac{1}{r} + \frac{1}{s} = 1$. *Then for every function* f *in* $L^r(\mathbb{R})$*, there holds*

$$\left(\int_{\mathbb{R}} |\mathcal{Q}_K\{f\}(\omega)|^s \, d\omega \right)^{\frac{1}{s}} \leq b^{-\frac{1}{s}} r^{\frac{1}{2r}} s^{-\frac{1}{2s}} \left(\int_{\mathbb{R}} |f(x)|^r \, dx \right)^{\frac{1}{r}}. \tag{4.2}$$

Proof. Applying the sharp Hausdorff-Young inequality related to the conventional Fourier transform results in

$$\left(\int_{\mathbb{R}} |\mathcal{F}\{f\}(\omega)|^s \, d\omega \right)^{1/s} \leq r^{\frac{1}{2r}} s^{-\frac{1}{2s}} \left(\int_{\mathbb{R}} |f(x)|^r \, dx \right)^{1/r}.$$

Replacing $f(t)$ by $g(t)$ defined by (3.1) results in

$$\left(\int_{\mathbb{R}} |\mathcal{F}\{g\}(\omega)|^s \, d\omega \right)^{1/s} \leq r^{\frac{1}{2r}} s^{-\frac{1}{2s}} \left(\int_{\mathbb{R}} |g(x)|^r \, dx \right)^{1/r}.$$

Setting $\omega = \omega b$ yields

$$\left(\int_{\mathbb{R}} |\mathcal{F}\{g\}(\omega b)\,|^s \, d\,(\omega b) \right)^{\frac{1}{s}} \leq r^{\frac{1}{2r}} s^{-\frac{1}{2s}} \left(\int_{\mathbb{R}} |g(x)|^r \, dx \right)^{\frac{1}{r}}. \tag{4.3}$$

Applying (3.1) and (3.2) to equation (4.3), yields

$$b^{\frac{1}{s}} \left(\int_{\mathbb{R}} \left| \mathcal{Q}_K\{f\}(\omega) e^{i(c\omega^2 + e\omega)} \right|^s \, d\omega \right)^{\frac{1}{s}} \leq r^{\frac{1}{2r}} s^{-\frac{1}{2s}} \left(\int_{\mathbb{R}} \left| f(x) e^{-i(ax^2 + dx)} \right|^r \, dx \right)^{\frac{1}{r}}.$$

Hence,

$$\left(\int_{\mathbb{R}} |\mathcal{Q}_K\{f\}(\omega)|^s \, d\omega \right)^{\frac{1}{s}} \leq b^{-\frac{1}{s}} r^{\frac{1}{2r}} s^{-\frac{1}{2s}} \left(\int_{\mathbb{R}} |f(x)|^r \, dx \right)^{\frac{1}{r}},$$

and proof is complete. ∎

4.3. Matolcsi-Szucs Inequality

Based on Housdorf-Young inequality for the quadratic-phase Fourier Transform, we obtain Matolcsi-Szucs inequality related to the quadratic- phase Fourier transform.

Theorem 4.3 (Matolcsi-Szucs Inequality). *Let* $f \in L^{r_1}(\mathbb{R}) \cap L^{r_2}(\mathbb{R})$ *such that* $1 < r_1 \leq r_2 \leq 2$, *then*

$$\|\mathcal{Q}_K\{f\}\|_{L^{s_2}(\mathbb{R})} \leq \left|\operatorname{supp}(\mathcal{Q}_K\{f\})\right|^{\frac{s_1 - s_2}{s_1 s_2}} b^{-\frac{1}{s_1}} r^{\frac{1}{2r_1}} s_1^{-\frac{1}{2s_1}} \left|\operatorname{supp}(f)\right|^{\frac{r_2 - r_1}{r_1 r_2}} \|f\|_{L^{r_2}(\mathbb{R})},$$

where $\frac{1}{r_1} + \frac{1}{s_1} = 1$ *and* $\frac{1}{r_2} + \frac{1}{s_2} = 1$.

Proof. Applying equation (4.2) and Hölder's inequality, we get

$$\begin{aligned} \|\mathcal{Q}_K\{f\}\|_{L^{s_2}(\mathbb{R})} &\leq \left|\operatorname{supp}(\mathcal{Q}_K\{f\})\right|^{\frac{s_1 - s_2}{s_1 s_2}} \|\mathcal{Q}_K\{f\}\|_{L^{s_1}(\mathbb{R})} \\ &\leq \left|\operatorname{supp}(\mathcal{Q}_K\{f\})\right|^{\frac{s_1 - s_2}{s_1 s_2}} b^{-\frac{1}{s_1}} r^{\frac{1}{2r_1}} s_1^{-\frac{1}{2s_1}} \|f\|_{L^{r_1}(\mathbb{R})}. \end{aligned} \tag{4.4}$$

Here $F = \mathrm{supp}(f)$. Similarly, we have

$$\begin{aligned}
\|f\|_{L^{r_1}(\mathbb{R})} &= \|\chi_F f\|_{L^{r_1}(\mathbb{R})} \\
&\leq \left(\int_{\mathbb{R}} |\chi_F(x)|\, dx\right)^{\frac{r_2-r_1}{r_1 r_2}} \left(\int_{\mathbb{R}} |f(x)|^{\frac{r_1 r_2}{r_1}}\, dx\right)^{\frac{r_1}{r_1 r_2}} \\
&= \left(\int_{\mathbb{R}} |\chi_F(x)|\, dx\right)^{\frac{r_2-r_1}{r_1 r_2}} \left(\int_{\mathbb{R}} |f(x)|^{r_2}\, dx\right)^{\frac{1}{r_2}} \\
&= \left|F\right|^{\frac{r_2-r_1}{r_1 r_2}} \|f\|_{L^{r_2}(\mathbb{R})} \\
&= \left|\mathrm{supp}(f)\right|^{\frac{r_2-r_1}{r_1 r_2}} \|f\|_{L^{r_2}(\mathbb{R})}. \qquad (4.5)
\end{aligned}$$

Including (4.5) into (4.4) gives

$$\|\mathcal{Q}_K\{f\}\|_{L^{s_2}(\mathbb{R})} \leq \left|\mathrm{supp}(\mathcal{Q}_K\{f\})\right|^{\frac{s_1-s_2}{s_1 s_2}} b^{-\frac{1}{s_1}} r^{\frac{1}{2r_1}} s_1^{-\frac{1}{2s_1}} \left|\mathrm{supp}(f)\right|^{\frac{r_2-r_1}{r_1 r_2}} \|f\|_{L^{r_2}(\mathbb{R})},$$

and the proof is complete. ■

5. Conclusion

In this paper, we have introduced the quadratic-phase Fourier transform. We have presented the relation between the one-dimensional quadratic-phase Fourier transform and the one-dimensional Fourier transform. We applied this relation to obtain several versions of the inequalities related to the proposed one-dimensional quadratic-phase Fourier transform.

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