Hilbert Journal of Mathematical Analysis
Volume 2 Number 1 (2023)
Pages 034–045

https://hilbertjma.org

ISSN 2985-7619

On the Fourier coefficients of the derivative with respect to celebrated orthogonal systems

Abdelhamid Rehouma[1,*]

[1] *Department of mathematics, Faculty of exact sciences, University Hama Lakhdar, Eloued, Algeria*
e-mail: rehoumaths@gmail.com

Abstract The main goal of this paper is to find the coefficients of finite linear combination of Jacobi polynomials and finite linear combination of the integrals of Legendre polynomials expansion of the derivative of a function in terms of the coefficients in the expansion of the original finite linear combination functions. More precisely, if Q_n is a sequence or orthogonal polynomials, and if

$$p(x) = \sum_{j=0}^{n} a_j Q_{n-j}(x)$$

is such that

$$p'(x) = \sum_{j=0}^{n-1} d_j Q_{n-j-1}(x).$$

We find an explicit relation for the coefficients d_j, as linear combinations of the coefficients a_j. This will be done for two celebrated classes of orthogonal functions, namely the Jacobi polynomials and the integrals of the Legendre polynomials.

MSC: 33C45; 42C05

Keywords: Orthogonal polynomials; Jacobi polynomials; Legendre polynomials; special functions; finite linear combination; Fourier coefficient; Three-term recurrence relation

Received: 08-12-2023 / Accepted: 21-02-2024 / Published: 28-02-2024
DOI : https://doi.org/10.62918/hjma.v2i1.19

1. Introduction

The need for approximating polynomials has been an important topic in many branches of mathematics and computer approximation. This is due to the easy nature of polynomials, which facilitates the treatment of other complicated functions.

Solving integrals, derivatives, ordinary and partial differential equations can be simplified extensively when polynomials replace other functions. Moreover, approximation formulas used in numerical methods are built mostly on polynomial approximations.

Thus, computers, calculators, and codes benefit from this approximation idea, where efficient algorithms can be implemented to find good approximations of some complicated problems.

*Corresponding author.

Published by Komunitas Analisis Matematika Indonesia.

It is due to the celebrated Weierstrass approximation theorem that any continuous function on a given interval $[a, b]$ can be uniformly approximated by polynomials.

However, orthogonal polynomials acquire the most attention due to their applications and easy computations, despite of their complicated forms in some cases. This being said, with the existing technology and built-in functions, the complicated nature of such polynomials becomes negligible.

In the sequel, a family $\{Q_n(x)\}_{n=0,1,2,\ldots}$ of nonzero polynomials is said to be orthogonal on the interval $[a, b]$, with respect to the density function $\omega(x)$ defined on the interval $[a, b]$, which is also called the weight function. Namely

$$\int_a^b Q_i(x)Q_j(x)\omega(x)dx = 0, \qquad i \neq j.$$

For simplicity, we will write $\{Q_n\}$ instead of $\{Q_n(x)\}_{n=0,1,2\ldots}$. In this paper, we assume that the degree of Q_n is n. This is the case in most applications. If $\omega(x) = 1$, we simply say that the set $\{Q_n\}$ is orthogonal, without referring to any weight.

Thus, given a polynomial φ_n, of degree n, it can be written as a linear combination

$$\varphi_n = \sum_{j=0}^{n} a_j Q_{n-j},$$

where

$$a_j = \frac{\int_a^b \varphi_n(x)Q_{n-j}(x)\omega(x)dx}{\int_a^b Q_{n-j}^2(x)\omega(x)dx}.$$

These coefficients are usually referred to as the Fourier coefficients of φ_n with respect to $\{Q_n\}$. Since φ_n is a polynomial of degree n, its derivative φ_n' is of degree at most $n-1$. Therefore,

$$\varphi_n' = \sum_{j=0}^{n-1} d_j Q_{n-j-1},$$

where

$$d_j = \frac{\int_a^b \varphi_n'(x)Q_{n-j-1}(x)\omega(x)dx}{\int_a^b Q_{n-j-1}^2(x)\omega(x)dx}.$$

The aim of this work is to find the coefficients of finite linear combination of Jacobi polynomials and the coefficients of finite linear combination of the integrals of Legendre polynomials expansion of the derivative of a function in terms of the coefficients in the expansion of the original finite linear combination functions. In particular, for $j = 1, 2, 3, \ldots, n-1$, we will be able to express the coefficients d_j as linear combination of a_j.

For more information and references, we refer to [1, 3–9] and also to the papers [10–14, 16] we refer to them for some relations and references.

2. The Jacobi polynomials case

Let Π_n be the space of polynomials of degree not greater than n. By $P_n^{(\alpha,\beta)}(x)$, where n is a non-negative whole number, we denote the n-th Jacobi polynomial. It is known that Jacobi polynomials with the same parameters α and β are orthogonal on $[-1, 1]$, with respect to to the Jacobi weight function $\omega^{\alpha,\beta}(x) = (1-x)^{\alpha}(1+x)^{\beta}$. The Jacobi weight function $\omega^{\alpha,\beta}$ belongs to $L^1([-1,1])$ if and only if $\alpha, \beta > -1$ (to be assumed throughout this section). Namely

$$\int_{-1}^{1} P_n^{(\alpha,\beta)}(x) P_m^{(\alpha,\beta)}(x) \omega^{\alpha,\beta}(x) dx = \gamma_{n,k}\delta_{n,m}, \tag{2.1}$$

where

$$\gamma_{n,k} = \left\| P_n^{(\alpha,\beta)} \right\|_{w^{\alpha,\beta}}^2 = \frac{2^{\alpha+\beta+1}\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{n!(2n+\alpha+\beta+1)\Gamma(n+\alpha+\beta+1)}. \tag{2.2}$$

Here, and in what follows, $\delta_{n,m}$ is the Kronecker delta. Denoting the Binomial coefficients by C_n^k, we shall need the following properties of Jacobi polynomials [15]:

$$P_n^{(\alpha,\beta)}(x) = \frac{\Gamma(n+\alpha+1)}{n!\Gamma(n+\alpha+\beta+1)} \sum_{k=0}^{n} C_k^n \frac{\Gamma(n+k+\alpha+\beta+1)}{\Gamma(k+\alpha+1)} \frac{(x-1)^k}{2^k}, \tag{2.3}$$

A recurrence relation for the derivative is given by

$$\frac{d}{dx} P_n^{(\alpha,\beta)}(x) = \frac{1}{2}(n+\alpha+\beta+1) P_{n-1}^{(\alpha+1,\beta+1)}(x). \tag{2.4}$$

Applying this formula recursively yields, for $k = 0, 1, 2, \ldots, n$,

$$\frac{d^k}{dx^k} P_n^{(\alpha,\beta)}(x) = \frac{1}{2^k} \frac{\Gamma(n+k+\alpha+\beta+1)}{\Gamma(n+\alpha+\beta+1)} P_{n-k}^{(\alpha+k,\beta+k)}(x). \tag{2.5}$$

Replacing x by $-x$ in (2.3) immediately leads to the symmetric relation

$$P_n^{(\alpha,\beta)}(-x) = (-1)^n P_n^{(\beta,\alpha)}(x).$$

Moreover,

$$P_n^{(\alpha,\beta)}(1) = \frac{2^n (\alpha)_n (\alpha+\beta)_n}{(\alpha+\beta)_{2n}}, \tag{2.6}$$

and

$$P_n^{(\alpha,\beta)}(-1) = \frac{(-2)^n (\beta)_n (\alpha+\beta)_n}{(\alpha+\beta)_{2n}}, \tag{2.7}$$

where

$$(\alpha)_n = (\alpha+n)(\alpha+n-1)\ldots(\alpha+1).$$

Here, $(\alpha)_n$ is the Pochhammer symbol.

The Jacobi polynomials are generated by the three-term recurrence relation [8, 16]

$$P_{n+1}^{(\alpha,\beta)}(x) = \left(a_n^{\alpha,\beta} x - b_n^{\alpha,\beta}\right) P_n^{(\alpha,\beta)}(x) - c_n^{\alpha,\beta} P_{n-1}^{(\alpha,\beta)}(x), \tag{2.8}$$

where

$$a_n^{\alpha,\beta} = \frac{(2n+\alpha+\beta+1)(2n+\alpha+\beta+2)}{2(n+1)(n+\alpha+\beta+1)}, \tag{2.9}$$

$$b_n^{\alpha,\beta} = \frac{\left(-\alpha^2+\beta^2\right)(2n+\alpha+\beta+1)}{2(n+1)(n+\alpha+\beta+1)(2n+\alpha+\beta)}, \tag{2.10}$$

and

$$c_n^{\alpha,\beta} = \frac{(n+\alpha)(n+\beta)(2n+\alpha+\beta+2)}{(n+1)(n+\alpha+\beta+1)(2n+\alpha+\beta)}. \tag{2.11}$$

Moreover, the Jacobi polynomials can be written in terms of their derivatives as follows

$$P_n^{(\alpha,\beta)}(x) = \sum_{l=0}^{n} \lambda_{n,l}^{(\alpha,\beta)} P_l^{(\alpha,\beta)\prime}(x), \tag{2.12}$$

where the coefficients $\lambda_{n,l}^{(\alpha,\beta)}, l = 0,1,2,\dots,n$ can be computed in terms of n,α,β as follows

$$\lambda_{n,l}^{(\alpha,\beta)} = \alpha_{n,l}^{(\alpha,\beta)} \int_{-1}^{1} P_n^{(\alpha,\beta)}(x)\, P_{l-1}^{(\alpha+1,\beta+1)}(x)\, \omega^{\alpha+1,\beta+1}(x)\, dx\ , \quad l = 0,1,2,\dots,n,$$

in which

$$\alpha_{n,l}^{(\alpha,\beta)} = \frac{(n-1)!\,(l+\alpha+\beta+1)(2n+\alpha+\beta+1)(n+\alpha+\beta+2)\,\Gamma(n+\alpha+\beta)}{2^{\alpha+\beta+3}\Gamma(n+\alpha+1)\,\Gamma(n+\beta+1)}.$$

Here and hereafter, $P_{-1} = 0$ by convention. The relation (2.12) takes the following form [15]

$$P_n^{(\alpha,\beta)}(x) = q_n^{\alpha,\beta} P_{n-1}^{(\alpha,\beta)\prime}(x) + r_n^{\alpha,\beta} P_n^{(\alpha,\beta)\prime}(x) + s_n^{\alpha,\beta} P_{n+1}^{(\alpha,\beta)\prime}(x), \tag{2.13}$$

where

$$q_n^{\alpha,\beta} = -\frac{2(n+\alpha)(n+\beta)}{(n+\alpha+\beta)(2n+\alpha+\beta)(2n+\alpha+\beta+1)}, \tag{2.14}$$

$$r_n^{\alpha,\beta} = \frac{2(\alpha-\beta)}{(2n+\alpha+\beta)(2n+\alpha+\beta+2)}, \tag{2.15}$$

and

$$s_n^{\alpha,\beta} = \frac{2(n+\alpha+\beta+1)}{(2n+\alpha+\beta+1)(2n+\alpha+\beta+2)}. \tag{2.16}$$

The main result is established in the following theorem.

Theorem 2.1. *Let $\alpha,\beta > -1$ and let $P_n^{(\alpha,\beta)}(x)$ denote the Jacobi polynomials, as defined in (2.3). For $n \geq 1$, let p_n denote another polynomial such that*

$$p_n(x) = \sum_{j=0}^{n} a_j P_{n-j}^{\alpha,\beta}(x) \quad \textit{and} \quad p_n'(x) = \sum_{j=0}^{n-1} d_j P_{n-j-1}^{\alpha,\beta}(x). \tag{2.17}$$

Then

$$d_0 = a_0 \frac{(2n+\gamma-1)(2n+\gamma)}{2(n+\gamma)} \tag{2.18}$$

and

$$d_1 = a_1 \frac{(2n+\gamma-1)(2n+\gamma)}{2(n+\gamma-1)} - a_0 \frac{(\alpha-\beta)(2n+\gamma)(2n+\gamma-1)^2}{2(n+\gamma)(n+\gamma-1)(2n+\gamma-2)}. \tag{2.19}$$

where $\gamma = \alpha + \beta$. *For* $j = 2, \ldots, n-1$, *we have*

$$\begin{aligned} d_j = a_j &\frac{(2n-2j+\gamma-1)(2n-2j+\gamma)}{2(n-j+\gamma)} \\ &- d_{j-1}\frac{2(\alpha-\beta)(2n-2j+\gamma-1)(2n-2j+\gamma)}{2(n-j+\gamma)(2n-2j+\gamma)(2n-2j+\gamma+2)} \\ &+ d_{j-2}\frac{(n-j+\alpha+1)(n-j+\beta+1)(2n-2j+\gamma-1)(2n-2j+\gamma)}{(n-j+\gamma)(n-j+\gamma+1)(2n-2j+\gamma+1)(2n-2j+\gamma+2)}. \end{aligned} \tag{2.20}$$

Proof. By integrating (2.13), we obtain

$$\int P^{(\alpha,\beta)}_{n-j-1}(x)\,dx = q^{\alpha,\beta}_{n-j-1}P^{(\alpha,\beta)}_{n-j-2}(x) + r^{\alpha,\beta}_{n-j-1}P^{(\alpha,\beta)}_{n-j-1}(x) + s^{\alpha,\beta}_{n-j-1}P^{(\alpha,\beta)}_{n-j}(x) + C_j \tag{2.21}$$

where

$$q^{\alpha,\beta}_{n-j-1} = -\frac{2(n-j-1+\alpha)(n-j-1+\beta)}{(n-j-1+\alpha+\beta)(2n-2j-2+\alpha+\beta)(2n-2j+\alpha+\beta-1)}, \tag{2.22}$$

$$r^{\alpha,\beta}_{n-j-1} = \frac{2(\alpha-\beta)}{(2n-2j-2+\alpha+\beta)(2n-2j+\alpha+\beta)}, \tag{2.23}$$

$$s^{\alpha,\beta}_{n-j-1} = \frac{2(n+\alpha+\beta+1)}{(2n-2j+\alpha+\beta-1)(2n-2j+\alpha+\beta)}, \tag{2.24}$$

and C_j is a certain constant to be fixed latter.

On the other hand, by integrating (2.17), yields

$$\begin{aligned} p_n(x) &= \sum_{j=0}^{n-1} d_j \int P^{(\alpha,\beta)}_{n-j-1}(x)\,dx \\ &= \sum_{j=0}^{n-1} d_j \left(q^{\alpha,\beta}_{n-j-1}P^{(\alpha,\beta)}_{n-j-2}(x) + r^{\alpha,\beta}_{n-j-1}P^{(\alpha,\beta)}_{n-j-1}(x) + s^{\alpha,\beta}_{n-j-1}P^{(\alpha,\beta)}_{n-j}(x)\right) + C' \\ &= \left(\sum_{j=0}^{n-1} d_j q^{\alpha,\beta}_{n-j-1}P^{(\alpha,\beta)}_{n-j-2}(x) + d_j r^{\alpha,\beta}_{n-j-1}P^{(\alpha,\beta)}_{n-j-1}(x) + d_j s^{\alpha,\beta}_{n-j-1}P^{(\alpha,\beta)}_{n-j}(x)\right) + C' \\ &= d_{n-1}q^{\alpha,\beta}_0 P^{(\alpha,\beta)}_{-1}(x) + d_{n-2}q^{\alpha,\beta}_1 P^{(\alpha,\beta)}_0(x) + \sum_{j=0}^{n-3} d_j q^{\alpha,\beta}_{n-j-1}P^{(\alpha,\beta)}_{n-j-2}(x) \\ &\quad + d_0 r^{\alpha,\beta}_{n-1}P^{(\alpha,\beta)}_{n-1}(x) + d_{n-1}r^{\alpha,\beta}_0 P^{(\alpha,\beta)}_0(x) + \sum_{j=1}^{n-2} d_j r^{\alpha,\beta}_{n-j-1}P^{(\alpha,\beta)}_{n-j-1}(x) \\ &\quad + d_0 s^{\alpha,\beta}_{n-1}P^{(\alpha,\beta)}_n(x) + d_1 s^{\alpha,\beta}_{n-2}P^{(\alpha,\beta)}_{n-1}(x) + \sum_{j=2}^{n-1} d_j s^{\alpha,\beta}_{n-j-1}P^{(\alpha,\beta)}_{n-j}(x) + C', \end{aligned}$$

where $C' = \sum_{j=0}^{n-1} C_j d_j$. This immediately implies

$$\begin{aligned}
p_n(x) &= d_{n-1} q_0^{\alpha,\beta} P_{-1}^{(\alpha,\beta)}(x) + d_{n-2} q_1^{\alpha,\beta} P_0^{(\alpha,\beta)}(x) + \sum_{j=2}^{n-1} d_{j-2} q_{n-j+1}^{\alpha,\beta} P_{n-j}^{(\alpha,\beta)}(x) \\
&\quad + d_0 r_{n-1}^{\alpha,\beta} P_{n-1}^{(\alpha,\beta)}(x) + d_{n-1} r_0^{\alpha,\beta} P_0^{(\alpha,\beta)}(x) + \sum_{j=2}^{n-1} d_{j-1} r_{n-j}^{\alpha,\beta} P_{n-j}^{(\alpha,\beta)}(x) \\
&\quad + d_0 s_{n-1}^{\alpha,\beta} P_n^{(\alpha,\beta)}(x) + d_1 s_{n-2}^{\alpha,\beta} P_{n-1}^{(\alpha,\beta)}(x) + \sum_{j=2}^{n-1} d_j s_{n-j-1}^{\alpha,\beta} P_{n-j}^{(\alpha,\beta)}(x) + C'.
\end{aligned}$$

That is,

$$\begin{aligned}
p_n(x) = &\sum_{j=2}^{n-1} \left(d_j s_{n-j-1}^{\alpha,\beta} + d_{j-1} r_{n-j}^{\alpha,\beta} + d_{j-2} q_{n-j+1}^{\alpha,\beta} \right) P_{n-j}^{(\alpha,\beta)}(x) \\
&+ \left(d_0 r_{n-1}^{\alpha,\beta} + d_1 s_{n-2}^{\alpha,\beta} \right) P_{n-1}^{(\alpha,\beta)}(x) + d_0 s_{n-1}^{\alpha,\beta} P_n^{(\alpha,\beta)}(x) \\
&+ \left(d_{n-1} r_0^{\alpha,\beta} + d_{n-2} q_1^{\alpha,\beta} + C' \right) P_0^{(\alpha,\beta)}(x) + d_{n-1} q_0^{\alpha,\beta} P_{-1}^{(\alpha,\beta)}(x).
\end{aligned}$$

Comparing this form with $p_n(x) = \sum_{j=0}^{n} a_j P_{n-j}^{\alpha,\beta}(x)$, and recalling that $P_{-1}^{(\alpha,\beta)}(x) = 0$, we obtain

$$a_0 = d_0 s_{n-1}^{\alpha,\beta} \tag{2.25}$$

$$a_1 = d_0 r_{n-1}^{\alpha,\beta} + d_1 s_{n-2}^{\alpha,\beta} \tag{2.26}$$

$$a_j = d_j s_{n-j-1}^{\alpha,\beta} + d_{j-1} r_{n-j}^{\alpha,\beta} + d_{j-2} q_{n-j+1}^{\alpha,\beta}, \qquad j = 2, \dots, n-1, \tag{2.27}$$

and

$$a_n = d_{n-1} r_0^{\alpha,\beta} + d_{n-2} q_1^{\alpha,\beta} + C'. \tag{2.28}$$

Substituting (2.14), (2.15) and (2.16) into (2.25), (2.26), (2.27) and (2.28) implies, for $j = 2, \dots, n-1$,

$$\begin{aligned}
a_0 &= d_0 \frac{2(n+\alpha+\beta)}{(2n+\alpha+\beta-1)(2n+\alpha+\beta)} \\
a_1 &= d_0 \frac{2(\alpha-\beta)}{(2n+\alpha+\beta-2)(2n+\alpha+\beta)} + d_1 \frac{2(n+\alpha+\beta-1)}{(2n+\alpha+\beta-1)(2n+\alpha+\beta)} \\
a_j &= d_j \frac{2(n-j+\alpha+\beta)}{(2n-2j+\alpha+\beta-1)(2n-2j+\alpha+\beta)} \\
&\quad + d_{j-1} \frac{2(\alpha-\beta)}{(2n-2j+\alpha+\beta)(2n-2j+\alpha+\beta+2)} \\
&\quad - d_{j-2} \frac{2(n-j+\alpha+1)(n-j+\beta+1)}{(n-j+\alpha+\beta+1)(2n-2j+\alpha+\beta+1)(2n-2j+\alpha+\beta+2)},
\end{aligned}$$

and

$$a_n = d_{n-1}\frac{2(\alpha-\beta)}{(\alpha+\beta)(\alpha+\beta+2)} - d_{n-2}\frac{2(\alpha+1)(\beta+1)}{(\alpha+\beta+1)(\alpha+\beta+2)(\alpha+\beta+3)} + C'. \tag{2.29}$$

Solving the first three equations for d_0, d_1 and d_j when $j = 2, \ldots, n-1$ implies

$$d_0 = a_0\frac{(2n+\alpha+\beta-1)(2n+\alpha+\beta)}{2(n+\alpha+\beta)},$$

$$d_1 = a_1\frac{(2n+\alpha+\beta-1)(2n+\alpha+\beta)}{2(n+\alpha+\beta-1)} - d_0\frac{(\alpha-\beta)(2n+\alpha+\beta-1)}{(n+\alpha+\beta-1)(2n+\alpha+\beta-2)}$$

and

$$\begin{aligned} d_j &= a_j\frac{(2n-2j+\gamma-1)(2n-2j+\gamma)}{2(n-j+\gamma)} \\ &\quad - d_{j-1}\frac{2(\alpha-\beta)(2n-2j+\gamma-1)(2n-2j+\gamma)}{2(n-j+\gamma)(2n-2j+\gamma)(2n-2j+\gamma+2)} \\ &\quad + d_{j-2}\frac{(n-j+\alpha+1)(n-j+\beta+1)(2n-2j+\gamma-1)(2n-2j+\gamma)}{(n-j+\gamma)(n-j+\gamma+1)(2n-2j+\gamma+1)(2n-2j+\gamma+2)}, \end{aligned}$$

where $\gamma = \alpha+\beta$. Notice that this gives all values for d_j when $j = 0, \ldots, n-1$. Now, for all j we my choose C_j, the constant of integration that appeared in (2.21) so that $C' := \sum_{j=0}^{n-1} C_j d_j$ satisfies (2.29). This completes the proof of the theorem. ∎

Remark 2.2. In the proof of Theorem 2.1, we had to choose certain values for C_j so that $\sum_{j=0}^{n-1} C_j d_j = L$, where

$$L = a_n - \left(d_{n-1}\frac{2(\alpha-\beta)}{(\alpha+\beta)(\alpha+\beta+2)} - d_{n-2}\frac{2(\alpha+1)(\beta+1)}{(\alpha+\beta+1)(\alpha+\beta+2)(\alpha+\beta+3)}\right).$$

We point out that this is always doable. Indeed, let j_0 be any index between 1 and $n-1$ such that $d_{j_0} \neq 0$. Then, we may let $C_{j_0} = \frac{L}{d_{j_0}}$ and $c_j = 0, j \neq j_0$. The existence of an index j_0 such that $d_{j_0} \neq 0$ is justified by the fact that $p' \not\equiv 0$.

3. Integral Legendre polynomials case

Let $(L_n)_{n=0,1,2,3,\ldots}$ be the Legendre polynomials. It is well know that the n-th Legendre polynomials, L_n satisfies the orthogonality relation

$$\int_{-1}^{1} L_n(x) L_m(x)\, dx = \frac{2}{2n+1}\delta_{n,m},$$

where $\delta_{n,m}$ is the Kronecker delta. For $n = 0, 1, \ldots,$ let $\{L_n(x)\}$ denotes the sequence of Legendre polynomials. The Legendre polynomials obeys a three-term recurrence relation:

$$L_0(x) = 1, \qquad L_1(x) = x$$

$$(n+1)L_{n+1}(x) = (2n+1)xL_n(x) - nL_{n-1}(x), \qquad n = 1, 2, \ldots$$

and the differential equation

$$\left(\left(1-x^2\right)L_n'(x)\right)' = -n(n+1)L_n(x). \tag{3.1}$$

In fact, the derivatives of the Legendre polynomials are also orthogonal with respect to the weight function $\omega(x) = 1 - x^2$, where we have

$$\int_{-1}^{1} L'_n(x) L'_m(x) \left(1 - x^2\right) dx = \frac{2n(n+1)}{2n+1} \delta_{n,m}. \tag{3.2}$$

In [2], polynomials that have roots at $x = \pm 1$, with all other roots in $(-1, 1)$ were treated and connected to the Legendre polynomials. Also, in [2], a discussion of polynomials whose inflection points and roots in $(-1, 1)$ coincide. These polynomials were referred to as **PIPCIR**. In [2], it was shown that these polynomials have the form

$$Q_n(x) = -\int_{x}^{1} L_{n-1}(t)\, dt \qquad -1 \leq x \leq 1, n \geq 2. \tag{3.3}$$

When $n = 0, 1$, we define $Q_0(x) = 1, Q_1(x) = x$. However these functions are not **PIPCIR** functions. It follows immediately from (3.3) that

$$Q'_n(x) = L_{n-1}(x) \text{ and } Q''_n(x) = L'_{n-1}(x), \qquad n \geq 1.$$

The functions $Q_n(x)$ and $Q_m(x)$ $(n, m \geq 2$ and $n \neq m)$ are orthogonal with respect to the weight function $\omega(x) = \dfrac{1}{1 - x^2}$. More precisely [2]

$$\int_{-1}^{1} \frac{Q_n(x) Q_m(x)}{1 - x^2} dx = \frac{2}{n(n-1)(2n-1)} \delta_{n,m}, \qquad n, m \geq 2.$$

Some important properties of the sequence $\{Q_n\}$ can be listed as follows [2]:

$$Q_n(x) = \left(1 - x^2\right) q_{n-2}(x), \qquad n \geq 2.$$

$$Q''_n(x) = -n(n-1) q_{n-2}(x), \qquad n \geq 2.$$

$$\left(1 - x^2\right) Q''_n(x) + n(n-1) Q_n(x) = 0, \qquad n \geq 1.$$

$$-2x Q''_n(x) + \left(1 - x^2\right) Q'''_n(x) + n(n-1) Q'_n(x) = 0, \qquad n \geq 1.$$

$$Q_n(\pm 1) = 0, \qquad n \geq 2.$$

$$Q'_n(1) = 1, \qquad n \geq 2.$$

$$Q''_n(1) = \frac{1}{2} (\pm 1)^{n-1} n(n-1), \qquad n \geq 2.$$

Orthogonality of the $\{Q_n\}$ with respect to the weight function $\frac{1}{1-x^2}$ implies orthogonality of $\{q_n\}$ with respect to the weight function $\left(1 - x^2\right)$. Indeed, for $n, m = 0, 1, \ldots,$ we have

$$\int_{-1}^{1} q_n(x) q_m(x) \left(1 - x^2\right) dx = \frac{2}{(n+2)(n+1)(2n+3)} \delta_{n,m}.$$

From this, it follows that

$$\| q_n(x) \|^2 = \int_{-1}^{1} q_n^2(x) \left(1 - x^2\right) dx = \frac{2}{(n+2)(n+1)(2n+3)}. \tag{3.4}$$

The recurrence relation

$$L_n(x) = \frac{1}{2n+1}\left(L'_{n+1}(x) - L'_{n-1}(x)\right)$$

for the Legendre polynomials can be stated equivalently in the form

$$Q'_n(x) = \frac{1}{2n-1}\left(Q''_{n+1}(x) - Q''_{n-1}(x)\right). \tag{3.5}$$

Integrating both sides of (3.5) yields

$$Q_n(x) = \frac{1}{2n-1}\left(Q'_{n+1}(x) - Q'_{n-1}(x)\right), \qquad n \geq 1. \tag{3.6}$$

We refer the reader to the informative reference [2] for further properties of $\{Q_n\}$, and its deep relations with the Legendre and Jacobi polynomials.

Due to the significance of these functions, the $\{Q_n\}$, we discuss the problem of finding the coefficients in the linear combination of the derivative of a function in terms of the coefficients for the original function. The following result is the main result in this regard.

Theorem 3.1. *Let $Q_n(x)$ denote the polynomials discussed above. For $n \geq 5$, let G_n be another polynomial such that*

$$G_n(x) = \sum_{j=0}^{n} \lambda_j Q_{n-j}(x) \quad \textit{and} \quad G'_n(x) = \sum_{j=0}^{n-1} v_j Q_{n-j-1}(x). \tag{3.7}$$

Then

$$\begin{aligned}
\nu_0 &= (2n-3)\lambda_0,\\
\nu_1 &= (2n-5)\lambda_1,\\
\nu_j &= \frac{2n-2j-3}{2n-2j+1}\nu_{j-2} + (2n-2j-3)\lambda_j, \quad 2 \leq j \leq n-3,\\
\nu_{n-2} &= \lambda_{n-2} - \frac{1}{5}\nu_{n-4},\\
\nu_{n-1} &= \lambda_{n-1} - \frac{1}{3}\nu_{n-3}.
\end{aligned}$$

Proof. By integrating both sides of (3.6) and noting, we get

$$\int Q_{n-j-1}(t)\,dt = \frac{1}{2n-2j-3}\left(Q_{n-j}(x) - Q_{n-j-2}(x)\right) + C_j,\ j \leq n-3, \tag{3.8}$$

where C_j is a certain constant. When

$$\begin{aligned}
&j = n-2, \text{we have} \int Q_{n-j-1}(t)\,dt = Q_2(x) + C_{n-2},\\
&j = n-1, \text{ we have} \int_{-1}^{x} Q_{n-j-1}(t)\,dt = x+1 = Q_1(x) + C_{n-1}.
\end{aligned} \tag{3.9}$$

Now integrating $G'_n(x) = \sum\limits_{j=0}^{n-1} v_j Q_{n-j-1}(x)$ implies

$$G_n(x) = \sum_{j=0}^{n-1} \nu_j \int Q_{n-j-1}(t)\,dt.$$

Using (3.8) and (3.9), we can write

$$\begin{aligned} G_n(x) = &\sum_{j=0}^{n-3} \nu_j \frac{1}{2n-2j-3} \left(Q_{n-j}(x) - Q_{n-j-2}(x)\right) \\ &+ \nu_{n-2} Q_2(x) + \nu_{n-1} Q_1(x) + C', \end{aligned}$$

where $C' = \sum_{j=0}^{n-1} C_j \nu_j$. This can be written as

$$\begin{aligned} G_n(x) &= \sum_{j=0}^{n-3} \nu_j \frac{1}{2n-2j-3} \left(Q_{n-j}(x) - Q_{n-j-2}(x)\right) \\ &\quad + \nu_{n-2} Q_2(x) + \nu_{n-1} Q_1(x) + C' \\ &= \sum_{j=0}^{n-3} \nu_j \frac{1}{2n-2j-3} Q_{n-j}(x) - \sum_{j=0}^{n-3} \nu_j \frac{1}{2n-2j-3} Q_{n-j-2}(x) \\ &\quad + \nu_{n-2} Q_2(x) + \nu_{n-1} Q_1(x) + C' \\ &= \sum_{j=2}^{n-3} \left(\frac{1}{2n-2j-3} \nu_j - \frac{1}{2n-2j+1} \nu_{j-2} \right) Q_{n-j}(x) \\ &\quad + \nu_0 \frac{1}{2n-3} Q_n(x) + \nu_1 \frac{1}{2n-5} Q_{n-1}(x) + \frac{1}{5} \nu_{n-4} Q_2(x) \\ &\quad + \frac{1}{3} \nu_{n-3} Q_1(x) + \nu_{n-2} Q_2(x) + \nu_{n-1} Q_1(x) + C'. \end{aligned} \tag{3.10}$$

But $G_n(x) = \sum\limits_{j=0}^{n} \lambda_j Q_{n-j}(x)$. Equating the corresponding coefficients from this and (3.10) implies

$$\begin{aligned} \lambda_0 &= \frac{1}{2n-3} \nu_0, \\ \lambda_1 &= \frac{1}{2n-5} \nu_1, \\ \lambda_j &= \frac{1}{2n-2j-3} \nu_j - \frac{1}{2n-2j+1} \nu_{j-2}, \quad 2 \le j \le n-3, \\ \lambda_{n-2} &= \frac{1}{5} \nu_{n-4} + \nu_{n-2}, \\ \lambda_{n-1} &= \frac{1}{3} \nu_{n-3} + \nu_{n-1}, \\ \lambda_n &= C'. \end{aligned}$$

Conversely, we get

$$\begin{aligned}
\nu_0 &= (2n-3)\,\lambda_0,\\
\nu_1 &= (2n-5)\,\lambda_1,\\
\nu_j &= \frac{2n-2j-3}{2n-2j+1}\nu_{j-2} + (2n-2j-3)\,\lambda_j, \quad 2 \le j \le n-3,\\
\nu_{n-2} &= \lambda_{n-2} - \frac{1}{5}\nu_{n-4},\\
\nu_{n-1} &= \lambda_{n-1} - \frac{1}{3}\nu_{n-3}.
\end{aligned}$$

Now, having found all coefficients, we choose C_j to satisfy $\sum_{j=0}^{n-1} \nu_j C_j = \lambda_n$. This completes the proof of the theorem. ∎

Remark 3.2. In the proof of Theorem 3.1, we need to find C_j so that $\sum_{j=0}^{n-1} \nu_j C_j = \lambda_n$. This is always solvable. This can be done in the same way as in Remark 2.2.

References

[1] M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions*, 10th Ed., Dover, New York, 1972.

[2] R. Belinsky, Integrals of Legendre polynomials and solution of some partial differential equations, *J. Appl. Anal.*, **6**(2) (2000), 259–282.

[3] J. P. Boyd, *Chebyshev and Fourier spectral methods*, Dover, 2nd Ed., Mineola, 2001.

[4] K. Dilcher and K. B. Stolarsky, Resultants and Discriminants of Chebyshev and related polynomials, *Trans. Amer. Math. Soc.*, **357** (2004), 965–981.

[5] E. H. Doha, W. M. Abd-Elhameed and M.M. Alsuyuti, On using third and fourth kinds Chebyshev polynomials for solving the integrated forms of high odd-order linear boundary value problems, *J. Egypt. Math. Soc.*, **23** (2015), 397–405.

[6] M. Foupouagnigni, *On difference and differential equations for modifications of classical orthogonal polynomials*, Habilitation Thesis, Department of Mathematics and Computer Science, University of Kassel, Kassel, January 2006.

[7] L. C. Frenzen and R. Wong, A uniform asymptotic expansion of the Jacobi polynomials with error bounds, *Canad. J. Math.*, **37**(5) (1985), 979–1007

[8] J. Gishe, *A finite family of q-orthogonal polynomials and resultants of Chebyshev polynomials*, Graduate School Theses and Dissertations, University of South Florida.

[9] J. Gishe and M. Ismail, Resultants of Chebyshev Polynomials, *J. Anal. Appl.*, **27**(4) (2008), 499-508.

[10] I. S. Gradshteyn and I. M. Ryzhik, On derivatives of orthogonal polynomials II, *Bull. Am. Math. Soc.*, **42** (1940), 261-264.

[11] F. Grunbaum, Random walks and orthogonal polynomials: some challenges, Probability, Geometry and Integrable Systems, *MSRI Publications*, **55** (2007). Boyd,Dilc,Doha,Foupouan,Frenz,Jemal,Gish,Gradsh,Albert,Moha

[12] M. Jamel and M. Dehghan, *On rational classical orthogonal polynomials and their applications for explicit computation of inverse Laplace transforms*, Mathematical Problems in Engineering 2005:2 (2005), pp. 215–230.

[13] S. Karlin and J. McGregor, *Linear growth models with many types and multidimensional Hahn polynomials*, in Theory and applications of special functions, 261288, ed. R. A. Askey, Academic Press, New York, 1975.

[14] A. Rehouma and M. J. Atia, Study of an example of Markov chains involving Chebyshev polynomials, *Integral Transforms Spec. Funct.* https://doi.org/10.1080/10652469.2022.2098286.

[15] J. Shen, T. Tang, and L.-L. Wang, *Spectral methods: algorithms, analysis and applications*, Springer, 2011.

[16] G. Szego, *Orthogonal Polynomials*, 4th ed., American Mathematical Society Colloquium Publications, Amer. Math. Soc. **23** Providence, RI, 1978.