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Boundedness of some classical operators in Bourgain-Morrey spaces

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Abstract In this paper, we establish relationships between the norms of both Bourgain-Morrey spaces and Wiener amalgam spaces. Therefore, we take advantage of these relations to study the action on Bourgain-Morrey spaces of some classical operators such as maximal operators, Hardy operators, some sublinear operators and their commutators, and the Fourier transform. We also establish in Bourgain-Morrey spaces a norm equivalence of Riesz potential and fractional maximal function.

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1. Introduction

This paper investigates the action of some classical operators of harmonic analysis on a class of function spaces related to Morrey spaces, called Bourgain-Morrey spaces.

Recall that the classical Lebesgue space $L^q := L^q(\mathbb{R}^d)$, with $q \in [1, \infty]$, is defined to be the set of all measurable complex functions f on $\mathbb{R}^d$ such that

$$\|f\|_q := \left[\int_{\mathbb{R}^d} |f(x)|^q \, dx\right]^{\frac{1}{q}} < \infty$$

with the usual modification made when $q = \infty$. In what follows, $|E|$ and χ_E denote the Lebesgue measure and the characteristic function of any measurable set $E \subset \mathbb{R}^d$, respectively. L^q_{loc} denotes the set of all measurable complex functions f on $\mathbb{R}^d$ such that $f\chi_K \in L^q$ for any bounded measurable subset K of $\mathbb{R}^d$.

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For $1 \leq q, \alpha \leq \infty$, the classical Morrey space $\mathcal{M}_q^\alpha := \mathcal{M}_q^\alpha(\mathbb{R}^d)$ is defined as the set of all elements f of L^q_{loc} for which

$$\|f\|_{\mathcal{M}_q^\alpha} := \sup_{x\in\mathbb{R}^d,\, r>0} |Q(x,r)|^{\frac{1}{\alpha}-\frac{1}{q}} \left\|f\chi_{Q(x,r)}\right\|_q < \infty,$$

where

$$Q(x,r) = \prod_{j=1}^{d} \left[x_j - \frac{r}{2}, x_j + \frac{r}{2}\right) , \qquad x = (x_1, x_2, ..., x_d) \in \mathbb{R}^d \text{ and } 0 < r < \infty.$$

Morrey spaces were introduced in 1938 by C. Morrey [14] to study both the regularity problem of solutions for quasi-linear elliptic partial differential equations and the calculus of variations. Note that, for $1 \leq q \leq \alpha \leq \infty$, L^α is included in $\mathcal{M}_q^\alpha$ and the inclusion is proper when $q < \alpha < \infty$. Moreover, Morrey spaces describe the local regularity of functions more precisely than Lebesgue spaces. Because of this issue, various Morrey-type spaces well suited for easy use in harmonic analysis and specifically in the fields of partial differential equations have been introduced. Examples of such spaces include Bourgain-Morrey spaces.

Recall that the theory of Bourgain-Morrey spaces goes back to Bourgain [2], who considered a special case to study the Stein-Tomas estimate. Since then, their use has turned out fruitful in the study of Fourier restriction, multipliers problems, and partial differential equations, and in the proof of refinements of Strichartz inequality (see [10, 12, 13] and the references therein). They are defined as follows.

Definition 1.1. Let $1 \leq q, \alpha, p \leq \infty$. The Bourgain-Morrey space $\mathcal{M}_{q,p}^\alpha := \mathcal{M}_{q,p}^\alpha(\mathbb{R}^d)$ is defined as the set of all $f \in L^q_{\mathrm{loc}}$ for which

$$\|f\|_{\mathcal{M}_{q,p}^\alpha} := \left\| \left\{ |Q_{k,m}|^{\frac{1}{\alpha}-\frac{1}{q}} \left\|f\chi_{Q_{k,m}}\right\|_q \right\}_{(k,m)\in\mathbb{Z}^d\times\mathbb{Z}} \right\|_{\ell^p} \tag{1.1}$$

is finite. Here and thereafter the sets

$$Q_{k,m} = \prod_{j=1}^{d} \left[k_j 2^m, (k_j+1)\, 2^m\right), \quad k = (k_1, k_2, ..., k_d) \in \mathbb{Z}^d, \;\; m \in \mathbb{Z}$$

are the usual dyadic cubes of $\mathbb{R}^d$ and for any sequence $\{a_i\}_{i\in\mathbb{I}}$ included in $\mathbb{C}$, where $\mathbb{I}$ is a countable set and $\mathbb{C}$ is the set of complex numbers,

$$\|\{a_i\}_{i\in\mathbb{I}}\|_{\ell^p} := \begin{cases} \left(\sum_{i\in\mathbb{I}} |a_i|^p\right)^{\frac{1}{p}} & \text{if } p < \infty \\ \sup_{i\in\mathbb{I}} |a_i| & \text{if } p = \infty. \end{cases}$$

Let $1 \leq q, \alpha, p \leq \infty$. It is well known that the space $\mathcal{M}_{q,p}^\alpha$ is nontrivial if and only if $1 \leq q < \alpha < p < \infty$ or $1 \leq q \leq \alpha \leq p = \infty$. When $1 \leq q < \alpha < p \leq \infty$, L^α is properly included in $\mathcal{M}_{q,p}^\alpha$, which is a linear subspace of $\mathcal{M}_q^\alpha$. Actually, the following inclusion and equality relations hold true.

$$L^\alpha \subset \mathcal{M}_{q,p}^\alpha \subset \mathcal{M}_{q,p_1}^\alpha \subset \mathcal{M}_{q,\infty}^\alpha = \mathcal{M}_q^\alpha \;\; , \quad 1 \leq q < \alpha < p \leq p_1 \leq \infty. \tag{1.2}$$

Many useful results, well known for Lebesgue or Morrey spaces, have been extended to the setting of Bourgain-Morrey spaces (see [4, 10, 12] and the references therein).

In this paper, we study in Bourgain-Morrey spaces the action of classical operators such as maximal operators, Hardy operators, some sublinear operators and their commutators, and the Fourier transform. We also establish in these spaces a norm equivalence of Riesz potential and fractional maximal function. In doing so, we express the norm $\|\cdot\|_{\mathcal{M}^{\alpha}_{q,p}}$ as mixed-norms of the Wiener amalgam space (L^q, ℓ^p) (see Section 2 for its definition) and the sequence space ℓ^p.

The rest of the paper is organized as follows. Section 2 deals with some preliminaries on the norms of Wiener amalgam spaces and Bourgain-Morrey spaces. Section 3 is devoted to proving the boundedness of maximal operators, Hardy operators, some sublinear operators and their commutators, and the Fourier transform in Bourgain-Morrey spaces. In Section 4, we establish an equivalence between $\mathcal{M}^{\alpha}_{q,p}$-norms of both Riesz potential and fractional maximal function.

2. Preliminaries on norms

We make some conventions on notations used in this paper. For $1 \le s \le \infty$, s' denotes the conjugate exponent of s : $\frac{1}{s'} = 1 - \frac{1}{s}$ with the convention $\frac{1}{\infty} = 0$. We use c as a generic positive constant whose value may change with each appearance. The expression $A \lesssim B$ means that $A \le cB$ for some independent constant $c > 0$ and $A \approx B$ means $A \lesssim B \lesssim A$. $\mathbb{R}^d$ is equipped with its usual Hilbert space structure and the Euclidean norm of $x \in \mathbb{R}^d$ is denoted by $|x|$. Let ρ be an element of $(0, \infty)$ and $x \in \mathbb{R}^d$. $B(x, \rho)$ denotes the ball centered at x with radius ρ and $\lambda B := B(x, \lambda\rho)$ for all $\lambda > 0$. Let (q, p) be an element of $[1, \infty]^2$. We set

$$I^{\rho}_k = \prod_{j=1}^{d} [k_j\rho, (k_j + 1)\rho) \ , \qquad k = (k_1, k_2, ..., k_d) \in \mathbb{Z}^d$$

and

$${}_{\rho}\|f\|_{q,p} = \left\| \left\{ \left\| f\chi_{I^{\rho}_k} \right\|_q \right\}_{k \in \mathbb{Z}^d} \right\|_{\ell^p}.$$

The Wiener amalgam space (L^q, ℓ^p) is defined by

$$(L^q, \ell^p) = \left\{ f \in L^1_{\rm loc} \ : \ {}_1\|f\|_{q,p} < \infty \right\}. \tag{2.1}$$

We recall that N. Wiener has introduced Wiener amalgam spaces [17] since 1926. However, their systematic study began with the work of F. Holland [11] in 1975. The paper of Fournier and Stewart [7] is a very interesting survey on these spaces. It is well known that $((L^q, l^p), {}_1\|\cdot\|_{q,p})$ is a Banach linear subspace of $L^1_{\rm loc}$. It is also true that $\{{}_{\rho}\|\cdot\|_{q,p} : \rho \in (0, \infty)\}$ is a family of mutually equivalent norms on (L^q, ℓ^p). Note that we can consider a continuous summation instead of a discrete one. More precisely, in definition (2.1), we can replace ${}_1\|f\|_{q,p}$ by ${}_{\rho}\widetilde{\|f\|}_{q,p}$ defined as follows:

$${}_{\rho}\widetilde{\|f\|}_{q,p} = \left(\int_{\mathbb{R}^d} \left\| f\chi_{B(y,\rho)} \right\|_q^p dy \right)^{\frac{1}{p}}, \quad \rho > 0 \tag{2.2}$$

with the usual modification made when $p = \infty$. Actually, for $f \in L^1_{\rm loc}$ and $\rho > 0$, we have

$${}_{\rho}\widetilde{\|f\|}_{q,p} \approx \rho^{\frac{d}{p}} \ {}_{\rho}\|f\|_{q,p} \approx \rho^{\frac{d}{p}} \ {}_1\|f\|_{q,p}. \tag{2.3}$$

We shall establish other equivalences with the above norms. In order to do this, we first prove the following preparatory result.

Lemma 2.1. *Let (m, ρ) be an element of $\mathbb{Z} \times \mathbb{R}_+^*$ such that $2^m \leq \rho < 2^{m+1}$. Then the number of elements of the set $\left\{\ell \in \mathbb{Z}^d \,:\, I_\ell^{2^m} \cap I_k^\rho \neq \varnothing\right\}$ is at most 3^d and that of the set $\left\{k \in \mathbb{Z}^d \,:\, I_\ell^{2^m} \cap I_k^\rho \neq \varnothing\right\}$ is at most 2^d.*

Proof. Let $k = (k_1, k_2, \ldots, k_d)$ and $\ell = (\ell_1, \ell_2, \ldots, \ell_d)$ be two elements of $\mathbb{Z}^d$. We have

$$I_\ell^{2^m} \cap I_k^\rho \neq \varnothing \Longrightarrow \exists\, x \in \mathbb{R}^d : \forall\, j \in \{1, 2, \ldots, d\}, \left\{ \begin{array}{l} \ell_j 2^m \leq x_j < (\ell_j + 1)2^m \\ k_j \rho \leq x_j < (k_j + 1)\rho. \end{array} \right.$$

1) Fix j in $\{1, 2, \ldots, d\}$. Then there exists a unique element β of $\mathbb{Z}$ such that $(\beta - 1)2^m \leq k_j \rho < \beta\, 2^m$. Therefore, we have

$$\begin{aligned}(\beta - 1)2^m \leq k_j\rho \leq x_j &< (\ell_j + 1)2^m = \ell_j 2^m + 2^m \\ &\leq x_j + 2^m < k_j\rho + \rho + 2^m < \beta\, 2^m + 2^{m+1} + 2^m = (\beta + 3)2^m.\end{aligned}$$

Thus, we get $\beta - 1 < \ell_j + 1 < \beta + 3$ and therefore ℓ_j belongs to $\{\beta - 1, \beta, \beta + 1\}$. Hence

$$\left\{\ell \in \mathbb{Z}^d \,:\, I_\ell^{2^m} \cap I_k^\rho \neq \varnothing\right\} \subset \left\{\ell \in \mathbb{Z}^d \,:\, \ell_j \in \{\beta - 1, \beta, \beta + 1\}\right\}.$$

Consequently, for any fixed $k \in \mathbb{Z}^d$, the number of elements of $\left\{\ell \in \mathbb{Z}^d \,:\, I_\ell^{2^m} \cap I_k^\rho \neq \varnothing\right\}$ is at most 3^d.

2) Fix j in $\{1, 2, \ldots, d\}$. Then there exists a unique element λ of $\mathbb{Z}$ such that $(\lambda - 1)\rho \leq \ell_j 2^m < \lambda\,\rho$. Therefore, we have

$$(\lambda - 1)\rho \leq \ell_j 2^m \leq x_j < k_j\rho + \rho \leq x_j + \rho < (\ell_j + 1)2^m + \rho < \lambda\,\rho + 2\rho = (\lambda + 2)\rho.$$

Thus, we get $\lambda - 1 < k_j + 1 < \lambda + 2$ and therefore k_j belongs to $\{\lambda - 1, \lambda\}$. Hence

$$\left\{k \in \mathbb{Z}^d \,:\, I_\ell^{2^m} \cap I_k^\rho \neq \varnothing\right\} \subset \left\{k \in \mathbb{Z}^d \,:\, k_j \in \{\lambda - 1, \lambda\}\right\}.$$

Consequently, for any fixed $\ell \in \mathbb{Z}^d$, the number of elements of $\left\{k \in \mathbb{Z}^d :\, I_\ell^{2^m} \cap I_k^\rho \neq \varnothing\right\}$ is atmost 2^d. ■

As a consequence of Lemma 2.1, the following results hold true.

Proposition 2.2. *Let (q, p) be an element of $[1, \infty]^2$ and f be an element of L^1_{loc}. Assume that (m, ρ) is an element of $\mathbb{Z} \times \mathbb{R}_+^*$ such that $2^m \leq \rho < 2^{m+1}$. Then we have*

$$3^{-\frac{d}{p}} 2^{-\frac{d}{p'}}\, {}_{2^m}\|f\|_{q,p} \leq\, {}_\rho\|f\|_{q,p} \leq 3^{\frac{d}{p'}} 2^{\frac{d}{p}}\, {}_{2^m}\|f\|_{q,p} \tag{2.4}$$

and therefore, there exist two real numbers A and B such that

$$3^{-\frac{d}{p}} 2^{-\frac{d}{p'}}\, A\, {}_{2^m}\widetilde{\|f\|}_{q,p} \leq\, {}_\rho\widetilde{\|f\|}_{q,p} \leq 3^{\frac{d}{p'}} 4^{\frac{d}{p}}\, B\, {}_{2^m}\widetilde{\|f\|}_{q,p}. \tag{2.5}$$

Proof. We only consider the case when $p < \infty$ because the proof of the case $p = \infty$ is quite similar and hence we omit the details. We have, for $k \in \mathbb{Z}^d$,

$$\begin{aligned}\|f\chi_{I_k^\rho}\|_q &\leq \sum_{\ell \in \mathbb{Z}^d} \|f\chi_{I_k^\rho \cap I_\ell^{2^m}}\|_q \\ &\leq 3^{\frac{d}{p'}} \left(\sum_{\ell \in \mathbb{Z}^d} \|f\chi_{I_k^\rho \cap I_\ell^{2^m}}\|_q^p\right)^{\frac{1}{p}} \quad \text{(by Hölder inequality and Lemma 2.1).}\end{aligned}$$

Therefore, we have

$$
\begin{aligned}
{}_{\rho}\|f\|_{q,p} &= \left\| \left\{ \left\| f\chi_{I_k^{\rho}} \right\|_q \right\}_{k\in\mathbb{Z}^d} \right\|_{\ell^p} \leq 3^{\frac{d}{p'}} \left(\sum_{k\in\mathbb{Z}^d} \left(\sum_{\ell\in\mathbb{Z}^d} \| f\chi_{I_k^{\rho}\cap I_\ell^{2^m}} \|_q^p \right) \right)^{\frac{1}{p}} \\
&= 3^{\frac{d}{p'}} \left(\sum_{\ell\in\mathbb{Z}^d} \left(\sum_{k\in\mathbb{Z}^d} \| f\chi_{I_k^{\rho}\cap I_\ell^{2^m}} \|_q^p \right) \right)^{\frac{1}{p}} \\
&\leq 3^{\frac{d}{p'}} 2^{\frac{d}{p}} \left(\sum_{\ell\in\mathbb{Z}^d} \| f\chi_{I_\ell^{2^m}} \|_q^p \right)^{\frac{1}{p}} \text{ (by Lemma 2.1).} \\
&= 3^{\frac{d}{p'}} 2^{\frac{d}{p}} \, {}_{2^m}\|f\|_{q,p}. \qquad (*)
\end{aligned}
$$

Similarly to what precedes, we have

$$
\begin{aligned}
{}_{2^m}\|f\|_{q,p} &= \left\| \left\{ \left\| f\chi_{I_k^{2^m}} \right\|_q \right\}_{k\in\mathbb{Z}^d} \right\|_{\ell^p} \leq 2^{\frac{d}{p'}} \left(\sum_{k\in\mathbb{Z}^d} \left(\sum_{\ell\in\mathbb{Z}^d} \| f\chi_{I_\ell^{\rho}\cap I_k^{2^m}} \|_q^p \right) \right)^{\frac{1}{p}} \\
&= 2^{\frac{d}{p'}} \left(\sum_{\ell\in\mathbb{Z}^d} \left(\sum_{k\in\mathbb{Z}^d} \| f\chi_{I_\ell^{\rho}\cap I_k^{2^m}} \|_q^p \right) \right)^{\frac{1}{p}} \\
&\leq 2^{\frac{d}{p'}} 3^{\frac{d}{p}} \left(\sum_{\ell\in\mathbb{Z}^d} \| f\chi_{I_\ell^{\rho}} \|_q^p \right)^{\frac{1}{p}} \text{ (by Lemma 2.1).} \\
&= 2^{\frac{d}{p'}} 3^{\frac{d}{p}} \, {}_{\rho}\|f\|_{q,p}. \qquad (**)
\end{aligned}
$$

Combining $(*)$ and $(**)$, we get (2.4).

The hypothesis implies that $2^{\frac{dm}{p}} \leq \rho^{\frac{d}{p}} < 2^{\frac{d(m+1)}{p}}$. Then, we multiply the three members of these inequalities by that of (2.4), respectively, and so we obtain what follows:

$$
3^{-\frac{d}{p}} 2^{-\frac{d}{p'}} 2^{\frac{dm}{p}} \, {}_{2^m}\|f\|_{q,p} \leq \rho^{\frac{d}{p}} \, {}_{\rho}\|f\|_{q,p} \leq 3^{\frac{d}{p'}} 2^{\frac{d}{p}} 2^{\frac{d(m+1)}{p}} \, {}_{2^m}\|f\|_{q,p}.
$$

Therefore, by (2.3), there exist two real numbers A and B such that

$$
3^{-\frac{d}{p}} 2^{-\frac{d}{p'}} A \, {}_{2^m}\widetilde{\|f\|}_{q,p} \leq {}_{\rho}\widetilde{\|f\|}_{q,p} \leq 3^{\frac{d}{p'}} 4^{\frac{d}{p}} B \, {}_{2^m}\widetilde{\|f\|}_{q,p}.
$$

The proof is complete. ∎

Remark 2.3. Let $1 \leq q, \alpha, p \leq \infty$ and $f \in L^1_{\mathrm{loc}}$. Note that, for any $k \in \mathbb{Z}^d$ and $m \in \mathbb{Z}$, $Q_{k,m} = I_k^{2^m}$. Thus, if $p < \infty$, then we have

$$\begin{aligned}
&\left\| \left\{ |Q_{k,m}|^{\frac{1}{\alpha}-\frac{1}{q}} \left\| f\chi_{Q_{k,m}} \right\|_q \right\}_{(k,m)\in\mathbb{Z}^d\times\mathbb{Z}} \right\|_{\ell^p} \\
&= \left\| \left\{ 2^{dm\left(\frac{1}{\alpha}-\frac{1}{q}\right)} \left\| f\chi_{I_k^{2^m}} \right\|_q \right\}_{(k,m)\in\mathbb{Z}^d\times\mathbb{Z}} \right\|_{\ell^p} \\
&= \left(\sum_{m\in\mathbb{Z}} \sum_{k\in\mathbb{Z}^d} \left[2^{dm\left(\frac{1}{\alpha}-\frac{1}{q}\right)} \left\| f\chi_{I_k^{2^m}} \right\|_q \right]^p \right)^{\frac{1}{p}} = \left(\sum_{m\in\mathbb{Z}} 2^{dm\left(\frac{1}{\alpha}-\frac{1}{q}\right)p} \sum_{k\in\mathbb{Z}^d} \left\| f\chi_{I_k^{2^m}} \right\|_q^p \right)^{\frac{1}{p}} \\
&= \left(\sum_{m\in\mathbb{Z}} 2^{dm\left(\frac{1}{\alpha}-\frac{1}{q}\right)p} \left\| \left\{ \left\| f\chi_{I_k^{2^m}} \right\|_q \right\}_{k\in\mathbb{Z}^d} \right\|_{\ell^p}^p \right)^{\frac{1}{p}} = \left(\sum_{m\in\mathbb{Z}} \left[2^{dm\left(\frac{1}{\alpha}-\frac{1}{q}\right)} {}_{2^m}\|f\|_{q,p} \right]^p \right)^{\frac{1}{p}} \\
&= \left\| \left\{ 2^{dm\left(\frac{1}{\alpha}-\frac{1}{q}\right)} {}_{2^m}\|f\|_{q,p} \right\}_{m\in\mathbb{Z}} \right\|_{\ell^p}.
\end{aligned}$$

The case $p = \infty$ is obtained in a similar way.

From what precedes and the equality (1.1), we obtain the following new expression of the norm of Bourgain-Morrey spaces, which is a mixture of the norms of the Wiener amalgam space (L^q, ℓ^p) and the sequence space ℓ^p:

$$\|f\|_{\mathcal{M}^{\alpha}_{q,p}} = \left\| \left\{ 2^{dm\left(\frac{1}{\alpha}-\frac{1}{q}\right)} {}_{2^m}\|f\|_{q,p} \right\}_{m\in\mathbb{Z}} \right\|_{\ell^p}. \tag{2.6}$$

Moreover, (2.3) shows that the norm in (2.6) is equivalent to the following:

$$\widetilde{\|f\|}_{\mathcal{M}^{\alpha}_{q,p}} = \left\| \left\{ 2^{dm\left(\frac{1}{\alpha}-\frac{1}{q}-\frac{1}{p}\right)} {}_{2^m}\widetilde{\|f\|}_{q,p} \right\}_{m\in\mathbb{Z}} \right\|_{\ell^p}. \tag{2.7}$$

It clearly follows from (2.6) and/or (2.7) that the Bourgain-Morrey space $\mathcal{M}^{\alpha}_{q,p}$ is embedded in the Wiener amalgam space (L^q, ℓ^p).

3. Boundedness of operators

In this section, we shall take advantage of relationships of norms established in Section 2 to study the boundedness of some classical operators on Bourgain-Morrey spaces.

3.1. Maximal operators and Hardy operators

Let us recall that the Hardy-Littlewood maximal operator $\mathfrak{M}$ is defined by

$$\mathfrak{M}f(x) = \sup_{r>0} |B(x,r)|^{-1} \int_{B(x,r)} |f(y)| dy\,, \qquad x \in \mathbb{R}^d\,,\ f \in L^1_{\mathrm{loc}}.$$

This operator is one of the most important operators in harmonic analysis because it controls various other important operators. This is the case of the sharp maximal function $\mathfrak{M}^\sharp$ defined below.

$$\mathfrak{M}^\sharp f(x) = \sup_{r>0} |B(x,r)|^{-1} \int_{B(x,r)} |f(y) - f_{B(x,r)}| dy\,, \qquad x \in \mathbb{R}^d\,,\ f \in L^1_{\mathrm{loc}},$$

where $f_{B(x,r)}$ denotes the average over $B(x,r)$ of f, defined by

$$f_{B(x,r)} = |B(x,r)|^{-1} \int_{B(x,r)} |f(y)|dy.$$

A straightforward calculation shows that

$$\mathfrak{M}^\sharp f(x) \leq 2\, \mathfrak{M} f(x) \ , \qquad x \in \mathbb{R}^d \ , \ f \in L^1_{\mathrm{loc}}. \tag{3.1}$$

We also have the Hardy operator $\mathcal{H}$ defined by

$$\mathcal{H} f(x) = |x|^{-d} \int_{|y|<|x|} f(y)dy \ , \qquad x \in \mathbb{R}^d \ , \ f \in L^1_{\mathrm{loc}}.$$

Using the fact that $|x-y| < 2\,|x|$ in the above integral, the following pointwise estimate holds true:

$$\mathcal{H}(|f|)(x) \lesssim 2^d\, \mathfrak{M} f(x) \ , \qquad x \in \mathbb{R}^d \ , \ f \in L^1_{\mathrm{loc}}. \tag{3.2}$$

The boundedness of $\mathfrak{M}$ has been studied in various Banach spaces. The one dimensional case for Wiener amalgam spaces is contained in [1, 3]. In order to obtain the result in the case $d > 1$, one may follow, with minor changes, the arguments used in the proofs of Theorems 4.2 and 4.5 in [3].

Proposition 3.1. *Let $1 < q, p < \infty$. Then, for any element $f \in L^1_{\mathrm{loc}}$, we have*

$$_1\|\mathfrak{M} f\|_{q,p} \lesssim \ _1\|f\|_{q,p}. \tag{3.3}$$

As an immediate consequence of the above proposition, we obtain the boundedness of the Hardy-Littlewood maximal operator in Bourgain-Morrey spaces. Recall that this result has been also proved in [10]. But our proof is more simple than the one given there.

Proposition 3.2. *Let $1 < q \leq \alpha \leq p < \infty$. Then, for any element $f \in L^1_{\mathrm{loc}}$, we have*

$$\|\mathfrak{M} f\|_{\mathcal{M}^\alpha_{q,p}} \lesssim \ \|f\|_{\mathcal{M}^\alpha_{q,p}}. \tag{3.4}$$

Proof. If $\alpha = q$ or $\alpha = p$ then $\mathcal{M}^\alpha_{q,p} = \{0\}$ and therefore we have nothing to prove. Hence we suppose that $1 < q < \alpha < p < \infty$. The inequalities (2.3) and (3.3) imply that, for any $m \in \mathbb{Z}$,

$$2^{dm\left(\frac{1}{\alpha}-\frac{1}{q}\right)}\ _{2^m}\|\mathfrak{M} f\|_{q,p} \lesssim \ 2^{dm\left(\frac{1}{\alpha}-\frac{1}{q}\right)}\ _{2^m}\|f\|_{q,p}.$$

Therefore, taking the ℓ^p-norm of both sides with respect to m, we obtain (3.4) thanks to (2.6). ∎

Proposition 3.2 and the inequalities (3.1) and (3.2) lead to the following result.

Corollary 3.3. *Let $1 < q \leq \alpha \leq p < \infty$. Then the operators $\mathfrak{M}^\sharp$ and $\mathcal{H}$ are bounded in $\mathcal{M}^\alpha_{q,p}$.*

3.2. Sublinear operators and their commutators

In this subsection, we consider the sublinear operators $\mathbb{T}$ satisfying the condition

$$|\mathbb{T} f(x)| \leq C \int_{\mathbb{R}^d} \frac{|f(y)|}{|x-y|^d} dy \ , \quad x \notin \mathrm{supp}\ f, \tag{3.5}$$

for any $f \in L^1$ with compact support. We point out that the condition (3.5) was first introduced by Soria and Weiss [16]. This condition is satisfied by many operators such as the Hardy-Littlewood maximal operator, Calderón-Zygmund singular integral operators, Bochner-Riesz operators at the critical index, and C. Fefferman's singular multiplier. The following result generalizes [5, Theorem 2.1], and its proof follows, with some minor modifications, the arguments used there. The reader can also see the proof of [6, Theorem 4.5].

Theorem 3.4. *Let $1 < q \leq \alpha \leq p \leq \infty$. If $\mathbb{T}$ is a sublinear operator which is bounded on L^q and satisfies the condition (3.5), then $\mathbb{T}$ is also bounded on $\mathcal{M}^{\alpha}_{q,p}$.*

Proof. • If $\alpha = q$ or $\alpha = p < \infty$ then $\mathcal{M}^{\alpha}_{q,p} = \{0\}$ and therefore we have nothing to prove.
• The case $p = \infty$ is just [5, Theorem 2.1].
• Suppose that $1 < q < \alpha < p < \infty$. Fix $\rho > 0$, $x \in \mathbb{R}^d$ and set $B := B(x, \rho)$. Let f be any element of $\mathcal{M}^{\alpha}_{q,p}$. We have

$$f = f\chi_{2B} + \sum_{k=1}^{\infty} f\chi_{(2^{k+1}B)\setminus(2^k B)}.$$

By the sublinearity of $\mathbb{T}$ and the condition (3.5), we obtain

$$|\mathbb{T}f| \lesssim |\mathbb{T}(f\chi_{2B})| + \sum_{k=1}^{\infty} \left|2^{k+1}B\right|^{-1} \int_{2^{k+1}B} |f(y)|dy$$

and therefore, an application of Hölder's inequality leads to

$$|\mathbb{T}f| \lesssim |\mathbb{T}(f\chi_{2B})| + \sum_{k=1}^{\infty} \left|2^{k+1}B\right|^{-\frac{1}{q}} \|f\chi_{2^{k+1}B}\|_q.$$

Taking the L^q-norm of both sides on the ball B and using the boundedness of $\mathbb{T}$ on L^q, we get

$$\|(\mathbb{T}f)\chi_B\|_q \lesssim \|f\chi_{2B}\|_q + \sum_{k=1}^{\infty} \left(2^k\right)^{-\frac{d}{q}} \|f\chi_{2^{k+1}B}\|_q.$$

Therefore, taking the L^p-norm of both sides, we obtain

$${}_{\rho}\widetilde{\|\mathbb{T}f\|}_{q,p} \lesssim {}_{2\rho}\widetilde{\|f\|}_{q,p} + \sum_{k=1}^{\infty} \left(2^k\right)^{-\frac{d}{q}} {}_{2^{k+1}\rho}\widetilde{\|f\|}_{q,p}.$$

Let $m \in \mathbb{Z}$ such that $2^m \leq \rho < 2^{m+1}$. Therefore, by (2.5), we have

$${}_{2^m}\widetilde{\|\mathbb{T}f\|}_{q,p} \lesssim {}_{2^{m+1}}\widetilde{\|f\|}_{q,p} + \sum_{k=1}^{\infty} \left(2^k\right)^{-\frac{d}{q}} {}_{2^{m+k+1}\rho}\widetilde{\|f\|}_{q,p}.$$

Multiplying both sides of the above inequality by $2^{dm\left(\frac{1}{\alpha}-\frac{1}{q}-\frac{1}{p}\right)}$, we obtain

$$\begin{aligned} 2^{dm\left(\frac{1}{\alpha}-\frac{1}{q}-\frac{1}{p}\right)} {}_{2^m}\widetilde{\|\mathbb{T}f\|}_{q,p} &\lesssim 2^{d(m+1)\left(\frac{1}{\alpha}-\frac{1}{q}-\frac{1}{p}\right)} {}_{2^{m+1}}\widetilde{\|f\|}_{q,p} \\ &\quad + \sum_{k=1}^{\infty} \left(2^k\right)^{d\left(\frac{1}{p}-\frac{1}{\alpha}\right)} 2^{d(m+k+1)\left(\frac{1}{\alpha}-\frac{1}{q}-\frac{1}{p}\right)} {}_{2^{m+k+1}}\widetilde{\|f\|}_{q,p}. \end{aligned}$$

Therefore, taking the ℓ^p-norm of both sides with respect to m, (2.7) yields

$$\widetilde{\|\mathbb{T}f\|}_{\mathcal{M}^{\alpha}_{q,p}} \lesssim \left(1 + \sum_{k=1}^{\infty} \left(2^k\right)^{d\left(\frac{1}{p}-\frac{1}{\alpha}\right)}\right) \widetilde{\|f\|}_{\mathcal{M}^{\alpha}_{q,p}}.$$

This provides the desired result because the series on the right hand side converges. ∎

Let us recall that the space BMO consists of all functions $\mathfrak{b}$ in L^1_{loc} for which

$$\|\mathfrak{b}\|_{BMO} := \sup_{r>0,\, x\in\mathbb{R}^d} \frac{1}{|B(x,r)|}\int_{B(x,r)} \left|\mathfrak{b}(x) - \mathfrak{b}_{B(x,r)}\right| dx$$

is finite, where $\mathfrak{b}_{B(x,r)}$ denotes the average over $B(x,r)$ of $\mathfrak{b}$.

Let $\mathbb{T}$ be a linear operator and $\mathfrak{b} \in BMO$. We define the linear commutator $[\mathfrak{b}, \mathbb{T}]$ by

$$[\mathfrak{b}, \mathbb{T}]f(x) = \mathbb{T}(\mathfrak{b}f)(x) - \mathfrak{b}(x)\mathbb{T}f(x), \quad f \in L^1_{\text{loc}}, \ \ x \in \mathbb{R}^d.$$

The next result shows the boundedness on Bourgain-Morrey spaces of the above-defined linear commutator.

Theorem 3.5. *Assume that* $1 < q \leq \alpha \leq p \leq \infty$. *Let* $\mathbb{T}$ *be a linear operator and* $\mathfrak{b} \in BMO$. *If* $\mathbb{T}$ *satisfies the condition (3.5) and* $[\mathfrak{b}, \mathbb{T}]$ *is bounded on* L^q, *then* $[\mathfrak{b}, \mathbb{T}]$ *is also bounded on* $\mathcal{M}^{\alpha}_{q,p}$.

Proof. • If $\alpha = q$ or $\alpha = p < \infty$ then $\mathcal{M}^{\alpha}_{q,p} = \{0\}$ and therefore we have nothing to prove.
• The case $p = \infty$ is just [5, Theorem 2.2].
• Suppose that $1 < q < \alpha < p < \infty$. Let $\rho > 0$ and $x \in \mathbb{R}^d$. We set $B := B(x,\rho)$. Arguing as in the proof of [5, Theorem 2.2], we get

$$\left\|[\mathfrak{b}, \mathbb{T}]\chi_B\right\|_q \lesssim \left\|f\chi_{2B}\right\|_q + \sum_{k=1}^{\infty}\left(2^k\rho\right)^{-d}\left[\int_B\left(\int_{2^{k+1}\rho\, B}|\mathfrak{b}(y) - \mathfrak{b}(z)|\ |f(y)|dy\right)^q dz\right]^{\frac{1}{q}}.$$

Therefore, using the John-Nirenberg theorem on BMO-functions (see [9, Corollary 7.1.8]), we obtain

$$\left\|[\mathfrak{b}, \mathbb{T}]\chi_B\right\|_q \lesssim \left\|f\chi_{2B}\right\|_q + \|\mathfrak{b}\|_{BMO}\sum_{k=1}^{\infty}\left(2^k\right)^{-\frac{d}{q}}\left\|f\chi_{2^{k+1}B}\right\|_q.$$

We obtain the desired result thanks to the proof of Theorem 3.4.

■

3.3. Fourier Transform

We define the Fourier transform $\mathcal{F}$ on the Schwartz space $\mathcal{S} := \mathcal{S}(\mathbb{R}^d)$ of test functions by the formula

$$\mathcal{F}f(\xi) = (2\pi)^{-\frac{d}{2}}\int_{\mathbb{R}^d} f(x)\,\mathrm{e}^{-ix\cdot\xi}dx\,, \quad f \in \mathcal{S}, \quad \xi \in \mathbb{R}^d.$$

Recall that $\mathcal{F}$ has an extension by duality to the space of tempered distributions $\mathcal{S}'$ on $\mathbb{R}^d$, which is a linear operator also called Fourier transform and denoted by $\mathcal{F}$. It is well known that Wiener amalgam spaces are linear subspaces of $\mathcal{S}'$ and it has been proved (see [7, Theorem 2.8]) that, if $1 \leq q, p \leq 2$ and f is in (L^q, l^p) then

$$_1\left\|\mathcal{F}f\right\|_{p',q'} \leq c\ _1\|f\|_{q,p}. \tag{3.6}$$

An easy consequence of (3.6) reads as follows.

Theorem 3.6. *Let us assume that* $1 \leq q \leq \alpha \leq p \leq 2$ *such that* $\frac{1}{q} + \frac{1}{p} = \frac{2}{\alpha}$ *and* f *is an element of* $\mathcal{M}^{\alpha}_{q,p}$. *Then, we have*

$$\left\|\mathcal{F}f\right\|_{\mathcal{M}^{\alpha'}_{p',q'}} \leq c\,\|f\|_{\mathcal{M}^{\alpha}_{q,p}}. \tag{3.7}$$

Proof. For any $m \in \mathbb{Z}^d$, thanks to (2.3) and (3.6), we have

$$_{2^m}\|\mathcal{F}f\|_{p',q'} \leq c\, _{2^m}\|f\|_{q,p}.$$

Multiplying both sides of the above inequality by $2^{dm\left(\frac{1}{\alpha'}-\frac{1}{p'}\right)}$, we obtain

$$2^{dm\left(\frac{1}{\alpha'}-\frac{1}{p'}\right)}\, _{2^m}\|\mathcal{F}f\|_{p',q'} \leq c\, 2^{dm\left(\frac{1}{p}-\frac{1}{\alpha}\right)}\, _{2^m}\|f\|_{q,p}.$$

Note that the hypothesis $\frac{1}{q}+\frac{1}{p}=\frac{2}{\alpha}$ implies that $\frac{1}{p}-\frac{1}{\alpha}=\frac{1}{\alpha}-\frac{1}{q}$ and therefore

$$\left\|\left\{2^{dm\left(\frac{1}{\alpha'}-\frac{1}{p'}\right)}\, _{2^m}\|\mathcal{F}f\|_{p',q'}\right\}_{m\in\mathbb{Z}}\right\|_{\ell^{q'}} \leq c\left\|\left\{2^{dm\left(\frac{1}{\alpha}-\frac{1}{q}\right)}\, _{2^m}\|f\|_{q,p}\right\}_{m\in\mathbb{Z}}\right\|_{\ell^{q'}}.$$

Since $1 \leq q \leq 2$, we have $2 \leq q' \leq \infty$ and so $1 \leq p \leq q' \leq \infty$. Thus, we get

$$\left\|\left\{2^{dm\left(\frac{1}{\alpha'}-\frac{1}{p'}\right)}\, _{2^m}\|\mathcal{F}f\|_{p',q'}\right\}_{m\in\mathbb{Z}}\right\|_{\ell^{q'}} \leq c\left\|\left\{2^{dm\left(\frac{1}{\alpha}-\frac{1}{q}\right)}\, _{2^m}\|f\|_{q,p}\right\}_{m\in\mathbb{Z}}\right\|_{\ell^{p}}$$

and therefore (2.6) leads to

$$\|\mathcal{F}f\|_{\mathcal{M}^{\alpha'}_{p',q'}} \leq c\,\|f\|_{\mathcal{M}^{\alpha}_{q,p}}.$$

The proof is complete. ∎

4. Equivalence of norms of Riesz potential and fractional maximal function in Bourgain-Morrey spaces

Let $0 < \gamma < d$. The Riesz potential operator I_γ is defined by

$$I_\gamma f(x) = \int_{\mathbb{R}^d} \frac{f(y)}{|x-y|^{d-\gamma}}dy$$

when this integral makes sense. This operator is known to be closely related to the fractional maximal operator $\mathfrak{M}_\gamma$ defined on $L^1_{\rm loc}$ by

$$\mathfrak{M}_\gamma f(x) = \sup_{r>0} |B(x,r)|^{\frac{\gamma}{d}-1}\int_{B(x,r)} |f(y)|dy\ , \quad x \in \mathbb{R}^d.$$

The following pointwise control is well known:

$$\mathfrak{M}_\gamma f(x) \lesssim I_\gamma(|f|)(x)\ , \qquad x \in \mathbb{R}^d\ ,\ f \in L^1_{\rm loc}. \tag{4.1}$$

Note that the boundedness on Bourgain-Morrey spaces of the Riesz potential and the fractional maximal operator has been studied by Hatano et al. (see Theorem 4.4 and Corollary 4.5 in [10]). Furthermore, the first author proved in [4] an extension of the Hardy-Littlewood-Sobolev theorem to the setting of these spaces. In this section, we establish in Bourgain-Morrey spaces a norm equivalence of I_γ and $\mathfrak{M}_\gamma$ when we deal with non-negative measurable functions. Recall that analogous results have been obtained for Morrey spaces in [8] and for the so-called Fofana spaces $(L^q,\ell^p)^\alpha$ in [6]. Our result reads as follows.

Theorem 4.1. *Let $1 < q \leq \alpha \leq p \leq \infty$ with $q < \infty$ and $0 < \gamma < d$. Then, for any non-negative element of $L^1_{\rm loc}$, we have*

$$\|I_\gamma f\|_{\mathcal{M}^{\alpha}_{q,p}} \approx \|\mathfrak{M}_\gamma f\|_{\mathcal{M}^{\alpha}_{q,p}}. \tag{4.2}$$

Proof. In view of Inequality (4.1), we only need to show that

$$\|I_\gamma f\|_{\mathcal{M}^\alpha_{q,p}} \lesssim \|\mathfrak{M}_\gamma f\|_{\mathcal{M}^\alpha_{q,p}}.$$

• If $\alpha = q$ or $\alpha = p < \infty$, then $\mathcal{M}^\alpha_{q,p} = \{0\}$ and therefore, we have nothing to prove.
• The case $p = \infty$ is just [8, Corollary 5.4].
• Suppose that $1 < q < \alpha < p < \infty$. Let $\rho > 0$ and $x \in \mathbb{R}^d$. We set $B := B(x, \rho)$. According to [8, Theorem 1.10], we have

$$\|(I_\gamma f)\chi_B\|_q \approx \|(\mathfrak{M}_\gamma f)\chi_B\|_q + |B|^{\frac{1}{q}} \int_{\mathbb{R}^d\setminus B} \frac{f(y)}{|x-y|^{d-\gamma}} dy. \tag{4.3}$$

Furthermore, we have

$$\begin{aligned} |B|^{\frac{1}{q}} \int_{\mathbb{R}^d\setminus B} \frac{f(y)}{|x-y|^{d-\gamma}} dy &= |B|^{\frac{1}{q}} \sum_{k=0}^{\infty} \int_{2^k\rho \le |x-y| < 2^{k+1}\rho} \frac{f(y)}{|x-y|^{d-\gamma}} dy \\ &\lesssim \sum_{k=0}^{\infty} (2^k)^{-\frac{d}{q}} |B|^{\frac{1}{q}} |2^{k+1}B|^{-1+\frac{\gamma}{d}} \|f\chi_{2^{k+1}B}\|_1 . \end{aligned}$$

Therefore, [8, Lemma 2.9] implies that

$$|B|^{\frac{1}{q}} \int_{\mathbb{R}^d\setminus B} \frac{f(y)}{|x-y|^{d-\gamma}} dy \lesssim \sum_{k=0}^{\infty} (2^k)^{-\frac{d}{q}} \|(\mathfrak{M}_\gamma f)\chi_B\|_q . \tag{4.4}$$

Since the series on the right hand side of (4.4) converges, it follows from (4.3) that

$$\|(I_\gamma f)\chi_B\|_q \lesssim \|(\mathfrak{M}_\gamma f)\chi_B\|_q.$$

This immediately implies the following desired result

$$\|I_\gamma f\|_{\mathcal{M}^\alpha_{q,p}} \lesssim \|\mathfrak{M}_\gamma f\|_{\mathcal{M}^\alpha_{q,p}}.$$

This ends the proof. ∎

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