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Legendre-Fenchel Duality in m-Convexity

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Abstract The Legendre-Fenchel transform, which maps a function to its convex conjugate, provides a dual perspective that is fundamental in understanding optimization problems. In this work, we show that $m-$convex function in $n-$dimensional normed spaces can be viewed as the Legendre-Fenchel dual problem. By constructing an epi-graph of m-convexity in n-dimensional normed spaces, we obtained some properties including the Legendre transform. Particularly, we prove this for certain convex functions.

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1. Introduction

Convexity is a fundamental principle in both mathematics and physics, conceals a hidden duality that reveals deep links between principal and dual spaces. The essence of this concept is the transformative Legendre transform, which highlights the symmetries between convex functions and their conjugates [2, 12]. The utility of convexity is highlighted by its property of preserving inequalities, rendering it indispensable in fields such as optimization, economics, and others [4, 7].

Legendre transform has been harnessed to address complex problems across various disciplines. In optimizing diverse systems, from solving the original Hamilton–Jacobi–Bellman (HJB) equation [1] and in control theory to reducing the Cramer-Lundberg (C-L) risk model in actuarial science [13]. Particularly noteworthy is its role in tackling challenges like High Absolute Risk Aversion (HARA), a phenomenon prevalent in decision theory and economics. In real Banach spaces, duality can be used to obtain stability of epsilon-isometry[9]. In the field of elasticity, the displacement-traction issue in three-dimensional linearized elasticity can be regarded as Legendre-Fenchel dual problems[3].

For the general convex function, the duality Legendre-Fenchel problem has been shown in [10]. In this work, we investigate the necessary conditions for the duality Legendre-Fenchel of m-convex functions in n-dimensional spaces. Additionally, we provide examples of some convex functions to illustrate this. We examine the relationship between principal and dual problems within both $m-$convexity and convexity.

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2. Preliminaries and Basic Facts

In this part, we recall some notations, definitions, and preliminary facts which are needed throughout this work.

A convex set is denoted Ω and the lower limit of the convex function f is denoted j. The domain of function f is denoted D_f. The dual space of a normed vector space X is denoted by X^* and the bidual space of X is denoted X^{**}. Therefore, $m-$convex set (convex set with respect to $m \in [0,1]$) is denoted by Ω_m.

Definition 2.1 (Convex function, [6, 10, 11])**.** Let Ω be a convex set, the function $f = f(\boldsymbol{x})$ with $\boldsymbol{x} \in \Omega$ is said to be convex if $\boldsymbol{x}, \boldsymbol{y} \in \Omega$ and $\alpha \in [0,1]$ imply

$$f(\alpha \boldsymbol{x} + (1-\alpha)\boldsymbol{y}) \leq \alpha f(\boldsymbol{x}) + (1-\alpha) f(\boldsymbol{y}). \tag{2.1}$$

Definition 2.2 (Strongly convex function, [6, 8])**.** Let c be a positive real number. A function f is said to be strongly convex function with modulus c if

$$f(\alpha \boldsymbol{x} + (1-\alpha)\boldsymbol{y}) \leq \alpha f(\boldsymbol{x}) + (1-\alpha) f(\boldsymbol{y}) - \frac{c}{2}\alpha(1-\alpha)\left\|\boldsymbol{x} - \boldsymbol{y}\right\|^2, \tag{2.2}$$

with $\boldsymbol{x}, \boldsymbol{y} \in \Omega$, and $\alpha \in [0,1]$.

A convex function $f : \mathbb{R}^n \to (-\infty, +\infty]$ is *proper* if $\{\boldsymbol{x} \in \Omega; f(\boldsymbol{x}) < +\infty\} \neq \emptyset$. It is said to be *lower semi-continuous* if $\boldsymbol{x}_j \to \boldsymbol{x}^*$ imply $f(\boldsymbol{x}^*) \leq \liminf f(\boldsymbol{x}_j)$. The function f is said to be *coercive* if $\lim\limits_{|\boldsymbol{x}| \to +\infty} f(\boldsymbol{x}) = +\infty$, and is called bounded from below if $j = \inf\limits_{\boldsymbol{x} \in \mathbb{R}^n} f(\boldsymbol{x}) > -\infty$.

Theorem 2.3 ([10])**.** *Any proper, lower semi-continuous, coercive function attains a minimum if it is bounded from below.*

Let $f : \mathbb{R}^n \to (-\infty, +\infty]$ be a proper function. The Legendre-transform of f is the function $f^* : \mathbb{R}^{n*} \to (-\infty, +\infty]$, given by

$$f^*(\boldsymbol{\xi}) = \sup_{\boldsymbol{x} \in \mathbb{R}^n} \{\boldsymbol{x} \cdot \boldsymbol{\xi} - f(\boldsymbol{x})\}.$$

Theorem 2.4 ([10])**.** *If $f : \mathbb{R}^n \to (-\infty, +\infty]$ is proper, convex, and lower semi-continuous, then each $\boldsymbol{x}_0 \in \mathbb{R}^n$ and $-\infty < c_0 < f(\boldsymbol{x}_0)$ takes affine function on $\mathbb{R}^n$ denoted by $h(\boldsymbol{x}) = \boldsymbol{a} \cdot \boldsymbol{x} + b$ such that*

$$h(\boldsymbol{x}_0) = c_0, \quad h(\boldsymbol{x}) < f(\boldsymbol{x}), \quad \boldsymbol{x} \in \mathbb{R}^n.$$

Theorem 2.5 ([10])**.** *If $f : \mathbb{R}^n \to (-\infty, +\infty]$ is proper, convex, and lower semi-continuous, then so is f^*.*

Theorem 2.6 ([10])**.** *If $f : \mathbb{R}^n \to (-\infty, +\infty]$ is proper, convex, and lower semi-continuous, then*

$$f^{**}(\boldsymbol{x}) = \sup_{\boldsymbol{\xi}} \{\boldsymbol{\xi} \cdot \boldsymbol{x} - f^*(\boldsymbol{\xi})\} = f(\boldsymbol{x}). \tag{2.3}$$

3. Legendre transform for some convex function

Proposition 3.1. *D_f is a convex set if $f : \mathbb{R}^n \to (-\infty, +\infty]$ is a convex function.*

Proof. Since f is convex, for all $\boldsymbol{x}, \boldsymbol{y} \in D_f$ and $\alpha \in [0,1]$, the inequality (2.1) holds, and it follows that $\alpha \boldsymbol{x} + (1-\alpha)\boldsymbol{y} \in D_f$. Therefore, D_f is a convex set. ∎

Proposition 3.2. *Let $f : \mathbb{R}^n \to (-\infty, +\infty]$ be defined by $f(\boldsymbol{x}) = \boldsymbol{a} \cdot \boldsymbol{x} + c$, where $\boldsymbol{a} \in \mathbb{R}^n$ and $c \in \mathbb{R}$. The Legendre transform of f satisfies equation (2.3).*

Proof. If $\boldsymbol{x} \in D_f$, then the Legendre transform of f satisfies

$$f^*(\boldsymbol{\xi}) = \sup_{\boldsymbol{x} \in D_f} \{\boldsymbol{x} \cdot \boldsymbol{\xi} - f(\boldsymbol{x})\} \geq \boldsymbol{x} \cdot \boldsymbol{\xi} - f(\boldsymbol{x}),$$

which implies

$$\begin{aligned} f^{**}(\boldsymbol{x}) &= \sup_{\boldsymbol{\xi} \in D_{f^*}} \{\boldsymbol{x} \cdot \boldsymbol{\xi} - f^*(\boldsymbol{\xi})\} \\ &\leq \sup_{\boldsymbol{\xi} \in D_{f^*}} \{\boldsymbol{x} \cdot \boldsymbol{\xi} - (\boldsymbol{x} \cdot \boldsymbol{\xi} - f(\boldsymbol{x}))\} = f(\boldsymbol{x}). \end{aligned} \tag{3.1}$$

For the reverse inequality, we consider

$$f^*(\boldsymbol{\xi}) = \sup_{\boldsymbol{x} \in D_f} \{\boldsymbol{x} \cdot \boldsymbol{\xi} - f(\boldsymbol{x})\} = \sup_{\boldsymbol{x} \in D_f} \{\boldsymbol{x} \cdot \boldsymbol{\xi} - (\boldsymbol{a} \cdot \boldsymbol{x} + c)\}.$$

Taking $\boldsymbol{\xi} = \boldsymbol{a}$, we have

$$f^*(\boldsymbol{a}) = \sup_{\boldsymbol{x} \in D_f} \{\boldsymbol{x} \cdot \boldsymbol{a} - (\boldsymbol{a} \cdot \boldsymbol{x} + c)\} = -c. \tag{3.2}$$

From (3.2) and for all $\boldsymbol{x} \in D_f$, we obtain

$$f(\boldsymbol{x}) = \boldsymbol{a} \cdot \boldsymbol{x} + c = \boldsymbol{a} \cdot \boldsymbol{x} - f^*(\boldsymbol{a}) \leq \sup_{\boldsymbol{a} \in D_{f^*}} \{\boldsymbol{a} \cdot \boldsymbol{x} - f^*(\boldsymbol{a})\} = f^{**}(\boldsymbol{x}). \tag{3.3}$$

. ■

In [10], the *subdifferential* of f at $\boldsymbol{x} \in D_f$ denoted by $\partial f(\boldsymbol{x})$ indicates the set of $\boldsymbol{\xi}$ satisfying

$$f(\boldsymbol{y}) \geq f(\boldsymbol{x}) + \boldsymbol{\xi} \cdot (\boldsymbol{y} - \boldsymbol{x}), \quad \boldsymbol{y} \in \mathbb{R}^n. \tag{3.4}$$

Proposition 3.3. *Let $f : \mathbb{R}^n \to (-\infty, +\infty]$ be defined by $f(\boldsymbol{x}) = \boldsymbol{a} \cdot \boldsymbol{x} + c$, where $\boldsymbol{a} \in \mathbb{R}^n$ and $c \in \mathbb{R}$. If $\boldsymbol{\xi}_0 \in \partial f(\boldsymbol{x}_0)$, then the Legendre transform of f satisfies Fenchel's Identity*

$$f(\boldsymbol{x}_0) + f^*(\boldsymbol{\xi}_0) = \boldsymbol{x}_0 \cdot \boldsymbol{\xi_0}. \tag{3.5}$$

Proof. By the definition, $\boldsymbol{\xi_0} \in \partial(\boldsymbol{x}_0)$ we have

$$f(\boldsymbol{x}) \geq f(\boldsymbol{x}_0) + \boldsymbol{\xi_0} \cdot (\boldsymbol{x} - \boldsymbol{x}_0), \quad \boldsymbol{x} \in \mathbb{R}^n.$$

and, in particular, $f(\boldsymbol{x}_0) < +\infty$, then we can rewrite as

$$\begin{aligned} \boldsymbol{\xi}_0 \cdot \boldsymbol{x}_0 - f(\boldsymbol{x}_0) &\geq \boldsymbol{\xi}_0 \cdot \boldsymbol{x} - f(\boldsymbol{x}) \\ &= \sup_{x \in \mathbb{R}^n} \{\boldsymbol{\xi}_0 \cdot \boldsymbol{x} - f(\boldsymbol{x})\} = f^*(\boldsymbol{\xi}_0). \end{aligned}$$

Thus, we obtain (3.5). ■

Example 3.4. Let $A : R^n \to R^m$ be a matrix and $\boldsymbol{b} \in \mathbb{R}^m$. Given $f(\boldsymbol{x}) = \frac{1}{2} |A\boldsymbol{x} - \boldsymbol{b}|^2$, f satisfies (2.3).

Proof. If $x \in D_f$, then the Legendre transform of f satisfies the inequality (3.1). Thus, it is sufficiently proven otherwise.

$$\begin{aligned}
f^*(\boldsymbol{\xi}) &= \sup_{\boldsymbol{x}\in D_f} \{\boldsymbol{x}\cdot\boldsymbol{\xi} - f(\boldsymbol{x})\} \\
&= \sup_{\boldsymbol{x}\in D_f} \left\{\boldsymbol{x}\cdot\boldsymbol{\xi} - \frac{1}{2}\left|A\boldsymbol{x}-\boldsymbol{b}\right|^2\right\} \\
&\geq \boldsymbol{x}\cdot\boldsymbol{\xi} - \frac{1}{2}\left|A\boldsymbol{x}-\boldsymbol{b}\right|^2.
\end{aligned}$$

Then, we have

$$\frac{1}{2}\left|A\boldsymbol{x}-\boldsymbol{b}\right|^2 \geq \boldsymbol{x}\cdot\boldsymbol{\xi} - f^*(\boldsymbol{\xi}).$$

By taking supremum on the right side with respect to the $\boldsymbol{\xi}$, we obtain

$$f(\boldsymbol{x}) = \frac{1}{2}\left|A\boldsymbol{x}-\boldsymbol{b}\right|^2 = \sup_{\boldsymbol{\xi}\in D_{f^*}} \{\boldsymbol{x}\cdot\boldsymbol{\xi} - f^*(\boldsymbol{\xi})\} = f^{**}(\boldsymbol{x}).$$

■

In [10], for the general convex function, the relation between principal and dual Legendre-Fenchel has been introduced. Given a function $\psi : \mathbb{R}^n \times \mathbb{R}^m \to (-\infty, +\infty]$, let

$$\begin{aligned}
&(X) \quad \inf\{\psi(\boldsymbol{x},0) | \boldsymbol{x}\in\mathbb{R}^n\} \\
&(X^*) \quad \sup\{-\psi^*(0,\boldsymbol{q}) | \boldsymbol{q}\in\mathbb{R}^m\}.
\end{aligned}$$

The normed vector spaces X be the *principal* and the *dual* problem denotes by X^*. Here, the expression

$$\begin{aligned}
\psi^*(\boldsymbol{p},\boldsymbol{q}) &= \sup_{(\boldsymbol{x},\boldsymbol{y})\in\mathbb{R}^n\times\mathbb{R}^m} \{\boldsymbol{x}\cdot\boldsymbol{p} + \boldsymbol{y}\cdot\boldsymbol{q} - \psi(\boldsymbol{x},\boldsymbol{y})\} \\
&= \sup_{\boldsymbol{x}\in\mathbb{R}^n} \{\boldsymbol{x}\cdot\boldsymbol{p} + \sup_{y\in\mathbb{R}^m} (\boldsymbol{y}\cdot\boldsymbol{q} - \psi(\boldsymbol{x},\boldsymbol{y}))\},
\end{aligned}$$

represents the Legendre transformation of $\psi(\boldsymbol{x},\boldsymbol{y})$. Let $\Phi : \mathbb{R}^m \to (-\infty, +\infty]$ given by

$$\Phi(\boldsymbol{y}) = \inf_{\boldsymbol{x}\in\mathbb{R}^n} \{\psi(x,y)\},$$

is proper, convex, and lower semi-continuous. Then

$$\begin{aligned}
\psi^*(0,\boldsymbol{q}) &= \sup_{\boldsymbol{x},\boldsymbol{y}} \{\boldsymbol{y}\cdot\boldsymbol{q} - \psi(\boldsymbol{x},\boldsymbol{y})\} \\
&= \sup_{\boldsymbol{y}} \{\boldsymbol{y}\cdot\boldsymbol{q} - \Phi(\boldsymbol{y})\} = \Phi^*(\boldsymbol{q}),
\end{aligned}$$

it follows that

$$\begin{aligned}
\sup_{\boldsymbol{q}} \{-\psi^*(0,\boldsymbol{q})\} &= \sup_{\boldsymbol{q}} \{-\Phi^*(q)\} \\
&= \sup_{\boldsymbol{q}} \{0\cdot\boldsymbol{q} - \Phi^*(\boldsymbol{q})\} \\
&= \Phi^{**}(0) = \Phi(0) = \inf_{\boldsymbol{q}} \{\psi(\boldsymbol{x},0)\}.
\end{aligned}$$

Thus, the problem (X) and (X^*) have the same value known as the Kuhn-Tucker duality.

4. Legendre-Fenchel on $m-$convexity

Inequalities in $m-$convexity and necessary conditions for the Legendre transformation are utilized to derive the Legendre-Fenchel duality on $m-$convexity.

Definition 4.1 ([5]). Let $m \in [0,1]$. A function $f : \Omega_m \to [0,+\infty]$ is called $m-$convex function if for any $\boldsymbol{x}, \boldsymbol{y} \in \Omega_m$ and $\alpha \in [0,1]$, we have

$$f(\alpha \boldsymbol{x} + m(1-\alpha)\boldsymbol{y}) \leq \alpha f(\boldsymbol{x}) + m(1-\alpha) f(\boldsymbol{y}). \tag{4.1}$$

Lemma 4.2. *Let $m \in [0,1]$ and $\Omega_m \subset \mathbb{R}^n$ be a $m-$convex set. Then $f(\boldsymbol{x}) : \Omega_m \to \mathbb{R}$ is $m-$convex if and only if its epi-graph given by*

$$\operatorname{epi}(f) = \{(\boldsymbol{x}, y) \mid \boldsymbol{x} \in \Omega_m \subset \mathbb{R}^n, y \geq f(\boldsymbol{x})\} \subset \mathbb{R}^{n+1}, \tag{4.2}$$

is $m-$convex in $\mathbb{R}^{n+1}$.

Proof. Let $m \in [0,1]$. Since f is $m-$convex function for all $\boldsymbol{x}, \boldsymbol{y} \in \Omega_m$, $\alpha \in [0,1]$, we have

$$f(\alpha \boldsymbol{x} + m(1-\alpha)\boldsymbol{y}) \leq \alpha f(\boldsymbol{x}) + m(1-\alpha) f(\boldsymbol{y}).$$

For any $(\boldsymbol{x}_1, y_1), (\boldsymbol{x}_2, y_2) \in \operatorname{epi}(f)$ such that $f(\boldsymbol{x}_1) \leq y_1$ and $f(\boldsymbol{x}_2) \leq y_2$, we obtain

$$f(\alpha \boldsymbol{x}_1 + m(1-\alpha)\boldsymbol{x}_2) \leq \alpha f(\boldsymbol{x}_1) + m(1-\alpha) f(\boldsymbol{x}_2) \leq \alpha y_1 + m(1-\alpha) y_2. \tag{4.3}$$

From (4.3), we can conclude that

$$\begin{aligned} \alpha(\boldsymbol{x}_1, y_2) + m(1-\alpha)(\boldsymbol{x}_2, y_2) &= (\alpha \boldsymbol{x}_1, \alpha y_1) + (m(1-\alpha)\boldsymbol{x}_2, m(1-\alpha) y_2) \\ &= (\alpha \boldsymbol{x}_1 + m(1-\alpha)\boldsymbol{x}_2, \alpha y_1 + m(1-\alpha) y_2) \in \operatorname{epi}(f). \end{aligned}$$

■

Theorem 4.3. *Let $m \in [0,1]$. If $f : \Omega_m \to [0,+\infty]$ is proper, $m-$convex, and lower semi-continuous, then so is f^*.*

Proof. Let $m \in [0,1]$. For each $\boldsymbol{x} \in \Omega_m \neq 0$ take the affine function

$$\psi_x(\boldsymbol{\xi}) = \boldsymbol{x} \cdot \boldsymbol{\xi} - f(\boldsymbol{x}),$$

of which epigraph $\operatorname{epi}(\psi_x) = \{(\boldsymbol{\xi}, y) | \boldsymbol{\xi} \in \Omega_m^*, y \geq \psi_x(\boldsymbol{\xi})\}$. For any $(\boldsymbol{\xi}_1, y_1), (\boldsymbol{\xi}_2, y_2) \in \operatorname{epi}(\psi_x)$ such that $\psi_x(\boldsymbol{\xi}_1) \leq y_1$ and $\psi_x(\boldsymbol{\xi}_2) \leq y_2$, we obtain

$$\begin{aligned} \psi_x(\alpha \boldsymbol{\xi}_1 + m(1-\alpha)\boldsymbol{\xi_2}) &= \boldsymbol{x} \cdot (\alpha \boldsymbol{\xi}_1 + m(1-\alpha)\boldsymbol{\xi}_2) - f(\boldsymbol{x}) \\ &= \alpha \boldsymbol{x} \cdot \boldsymbol{\xi}_1 + m(1-\alpha)\boldsymbol{x} \cdot \boldsymbol{\xi}_2 - f(\boldsymbol{x}) \\ &= \alpha \psi_x(\boldsymbol{\xi}_1) + m(1-\alpha)\psi_x(\boldsymbol{\xi}_2) - (1-m)(1-\alpha) f(\boldsymbol{x}). \end{aligned}$$

Since $f(\boldsymbol{x}) \geq 0$ we have

$$\psi_x(\alpha \boldsymbol{\xi}_1 + m(1-\alpha)\boldsymbol{\xi_2}) \leq \alpha \psi_x(\boldsymbol{\xi}_1) + m(1-\alpha)\psi_x(\boldsymbol{\xi}_2).$$

Thus, ψ_x is a $m-$convex function and

$$\operatorname{epi}(f^*) = \bigcap_{x \in \Omega_m} \operatorname{epi}(\psi_x),$$

therefore, $f^*(\boldsymbol{\xi})$ is $m-$convex and lower semi-continuous. To demonstrate that $\Omega_m^* \neq 0$, consider $\boldsymbol{a} \in \Omega_m$ and $b \in \mathbb{R}$ such that

$$\boldsymbol{a} \cdot \boldsymbol{x} + b < f(\boldsymbol{x}).$$

Consequently,

$$f^*(\boldsymbol{\xi}) = \sup_{\boldsymbol{x}\in\Omega_m} \{\boldsymbol{x}\cdot\boldsymbol{\xi} - f(\boldsymbol{x})\} \leq \sup_{\boldsymbol{x}\in\Omega_m} \{(\boldsymbol{\xi}-\boldsymbol{a})\cdot\boldsymbol{x} - b\},$$

implies $f^*(\boldsymbol{a}) \leq -b$. ■

5. Concluding Remarks

In this paper, we introduce the Legendre-Fenchel transform for m-convex functions in n-dimensional spaces. We present examples of convex functions and establish the relationship between the principal and dual problems in both m-convexity and convexity. Specifically, we establish the necessary condition for an $m-$convex function, which is a function exhibiting $m-$convexity must be non-negative valued. Additionally, for $m = 1$, we encounter the original convexity problem, while for $m \in [0, 1)$, we deal with a general function that is not necessarily strictly convex. Further research could be conducted on the convolution of the functional equation in both the principal and dual problems.

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