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Embedding from Morrey spaces to Morrey-Stummel spaces

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Abstract In this paper, we study the relation between Stummel spaces, Morrey spaces, and Lebesgue spaces. We show the existence of an embedding from Lebesgue spaces to Stummel spaces, and also from Morrey spaces to Morrey-Stummel spaces. The key of showing the existence of those embeddings relies on the boundedness of Riesz potential operator both on Morrey spaces and Lebesgue spaces.

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1. Introduction

For $1 \leq p < \infty$ and $0 \leq \lambda \leq n$, the Morrey space $L^{p,\lambda} = L^{p,\lambda}(\mathbb{R}^n)$ is the set of all locally integrable functions f on $\mathbb{R}^n$ that satisfy $\|f\|_{L^{p,\lambda}} < \infty$, where

$$\|f\|_{L^{p,\lambda}} := \sup_{x\in\mathbb{R}^n,\ r>0} \left(\frac{1}{r^\lambda} \int_{B(x,r)} |f(y)|^p dy \right)^{\frac{1}{p}}.$$

Morrey spaces can be viewed as a generalization of Lebesgue spaces in the following way: we allow the integral $\int_{B(x,r)} |f(y)|^p dy$ over an arbitrary open ball $B(x,r)$ to grow with an order of r^λ as r grows to ∞. Note that when $\lambda = 0$, we have $L^{p,0} = L^p$, the usual Lebesgue space. Historically, these spaces were introduced by C.B. Morrey [6] to study the behavior of solutions to some partial differential equations. Indeed, the notion of Morrey spaces is important in the study of Navier-Stokes [4] and Schrödinger equations [5].

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Meanwhile, Stummel classes were first developed by Ragusa and Zamboni in 2001 [7] as the set $\mathfrak{S}_\alpha$ of all locally integrable function f on $\mathbb{R}^n$ that satisfy

$$\eta(r) := \sup_{x\in\mathbb{R}^n} \int\limits_{B(x,r)} \frac{|f(y)|}{|x-y|^{n-\alpha}} dy \to 0, \quad \text{as } r \to 0,$$

where $0 < \alpha < n$. They call $\eta(r)$ the *Stummel modulus* of f. Later, this Stummel classes were modified by Almeida and Samko [1] as the set of all measurable functions f that satisfy

$$\|f\|_{\mathfrak{S}^{p,\beta}} = \sup_{x\in\mathbb{R}^n} \left(\int\limits_{\mathbb{R}^n} \frac{|f(y)|^t}{|x-y|^\beta} \, dy \right)^{\frac{1}{t}} < \infty,$$

where $1 \le t < \infty$ and $0 < \beta < n$, and these classes thus become Stummel spaces $\mathfrak{S}^{t,\beta} = \mathfrak{S}^{t,\beta}(\mathbb{R}^n)$.

Inspired by the works of Ragusa-Zamboni and Almeida-Samko, we shall here introduce a more general type of Stummel spaces $\mathfrak{S}^{t,s,\beta} = \mathfrak{S}^{t,s,\beta}(\mathbb{R}^n)$ for $1 \le t < \infty,\ 1 \le s < \infty$, and $0 < \beta < n$, which consists of all measurable functions f that satisfy

$$\|f\|_{\mathfrak{S}^{t,s,\beta}} = \left(\int\limits_{\mathbb{R}^n} \left(\int\limits_{\mathbb{R}^n} \frac{|f(y)|^t}{|x-y|^\beta} \, dy \right)^{\frac{s}{t}} dx \right)^{\frac{1}{s}} < \infty. \tag{1.1}$$

Note that if we let $s \to \infty$, then we get $\mathfrak{S}^{t,\infty,\beta} = \mathfrak{S}^{t,\beta}$.

As one may observe, there is a close relation between Morrey spaces and Stummel spaces. It can be seen that, members of a Stummel space can be identified as members of a Morrey space (see [7]). In this paper, we shall show the existence of embedding from Lebesgue spaces to Stummel spaces, as well as from Morrey spaces to Morrey-Stummel spaces, which we shall discuss in the next section.

2. Preliminaries

Recall that for $1 \le p < \infty$, the Lebesgue space $L^p = L^p(\mathbb{R}^n)$ consists of all measurable functions f that satisfy

$$\|f\|_{L^p} = \left(\int\limits_{\mathbb{R}^n} |f(y)|^p \, dy \right)^{\frac{1}{p}} < \infty.$$

The boundedness of various operators on Lebesgue spaces have been studied extensively. One of the operators that is relevant with our purpose is the fractional integral operator or the Riesz potential I_α, where $0 < \alpha < n$, which is defined by

$$I_\alpha f(x) = \int\limits_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}} dy, \quad x \in \mathbb{R}^n,$$

for any suitable functions f on $\mathbb{R}^n$. The Riesz potential I_α is known to be bounded from L^p to L^q for $1 < p < q < \infty$ where $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$. The associated inequality that shows the boundedness of the operator is

$$\|I_\alpha f\|_{L^q} \le C_{p,q} \|f\|_{L^p}, \quad f \in L^p,$$

which is known as the Hardy-Littlewood-Sobolev inequality. This result is extended by Chiarenza and Frasca to Morrey spaces as in the following theorem.

Theorem 2.1. *[2] Let $0 < \alpha < n$, $1 < p < \frac{n}{\alpha}$, and $0 \le \lambda < n - \alpha p$, and let $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n-\lambda}$. Then there exists $C_{p,q} > 0$ such that*

$$||I_\alpha f||_{L^{q,\lambda}} \le C_{p,q}||f||_{L^{p,\lambda}}, \quad f \in L^{p,\lambda}.$$

This result has been extended further to generalized Morrey spaces $L^{p,\phi} = L^{p,\phi}(\mathbb{R}^n)$, which consists of all measurable functions f such that

$$\|f\|_{L^{p,\phi}} = \sup_{x\in\mathbb{R}^n, r>0} \frac{1}{\phi(r)} \left(\frac{1}{|B(x,r)|} \int\limits_{B(x,r)} |f(y)|^p \, dy \right)^{\frac{1}{p}} < \infty.$$

Here $\phi : (0, \infty) \to (0, \infty)$ satisfies the doubling condition, that is, there exists $C_1 > 0$ such that

$$\frac{1}{2} \le \frac{r_1}{r_2} \le 2 \Rightarrow \frac{1}{C_1} \le \frac{\phi(r_1)}{\phi(r_2)} \le C_1, \tag{2.1}$$

and the inequality

$$\int_R^\infty \frac{\phi^p(r)}{r} dr \le C_2 \phi^p(R), \tag{2.2}$$

for every $R > 0$. Note that if $\phi(r) \equiv r^{\frac{\lambda-n}{p}}$, then $L^{p,\phi} = L^{p,\lambda}$.

The following result is due to Gunawan and Eridani [3].

Theorem 2.2. *[3] Suppose that ϕ satisfies (2.1) and (2.2). If $\phi(r) \le Cr^\gamma$ for $-\frac{n}{p} \le \gamma < -\alpha$ and $1 < p < \frac{n}{\alpha}$, then we have*

$$\|I_\alpha f\|_{L^{q,\psi}} \le C\|f\|_{L^{p,\phi}}, \quad f \in L^{p,\phi},$$

where $q = \frac{\gamma p}{\alpha+\gamma}$ and $\psi(r) = \phi(r)^{\frac{p}{q}}$, $r > 0$.

The above theorem will be used to study the relation between generalized Morrey spaces and generalized Morrey-Stummel spaces, denoted by $\mathfrak{S}_\phi^{t,s,\beta} = \mathfrak{S}_\phi^{t,s,\beta}(\mathbb{R}^n)$, where $1 \le t < \infty$, $1 \le s < \infty$, and $0 < \beta < n$. Here a measurable function f belongs to the generalized Morrey-Stummel space $\mathfrak{S}_\phi^{t,s,\beta}$ if

$$\|f\|_{\mathfrak{S}_\phi^{t,s,\beta}} = \sup_{z\in\mathbb{R}^n, r>0} \frac{1}{\phi(r)} \left(\frac{1}{|B(z,r)|} \int\limits_{B(z,r)} \left(\int\limits_{\mathbb{R}^n} \frac{|f(y)|^t}{|x-y|^\beta} \, dy \right)^{\frac{s}{t}} dx \right)^{\frac{1}{s}} < \infty. \tag{2.3}$$

Note that when $\phi(r) \equiv r^{\frac{\lambda-n}{s}}$, we write $\mathfrak{S}_\phi^{t,s,\beta} = \mathfrak{S}_\lambda^{t,s,\beta}$, which we recognize as the (classical) Morrey-Stummel space. Here the norm is given by

$$\|f\|_{\mathfrak{S}_\lambda^{t,s,\beta}} = \sup_{z\in\mathbb{R}^n, r>0} \left(\frac{1}{r^\lambda} \int\limits_{B(z,r)} \left(\int\limits_{\mathbb{R}^n} \frac{|f(y)|^t}{|x-y|^\beta} \, dy \right)^{\frac{s}{t}} dx \right)^{\frac{1}{s}}.$$

In general, we still require ϕ to satisfy (2.1) and (2.2).

3. Main Results

In this part, we present our main results, which are the embedding between Lebesgue spaces and Stummel spaces, and the embedding between Morrey spaces and Stummel spaces. We start by establishing a theorem that shows the existence of embedding between Lebesgue and Stummel spaces.

Theorem 3.1. *Let $0 < \alpha < n$, $\alpha = \frac{n}{p} - \frac{n}{q}$, and $1 < p < q < \infty$. Suppose that $q = \frac{s}{t}$, where $1 \leq t < s < \infty$. Then we have $L^{pt}(\mathbb{R}^n) \subseteq \mathfrak{S}^{t,s,n-\alpha}(\mathbb{R}^n)$. Moreover, the identity map $I : L^{pt} \to \mathfrak{S}^{t,s,n-\alpha}$ is bounded.*

Proof. Let $f \in L^{pt}$, which is equivalent to $|f|^t \in L^p$. From Hardy-Littlewood-Sobolev inequality, we get the result that

$$\|I_\alpha |f|^t\|_{L^q} \leq C_{p,q} \||f|^t\|_{L^p}.$$

Since

$$\||f|^t\|_{L^p} = \|f\|^t_{L^{pt}},$$

we obtain

$$\|I_\alpha |f|^t\|_{L^q} \leq C_{p,q} \|f\|^t_{L^{pt}}.$$

Therefore

$$\|I_\alpha |f|^t\|^{\frac{1}{t}}_{L^q} \leq C'_{p,q} \|f\|_{L^{pt}}. \tag{3.1}$$

Now observe that, since $q = \frac{s}{t}$, we have

$$\|I_\alpha |f|^t\|^{\frac{1}{t}}_{L^q} = \|I_\alpha |f|^t\|^{\frac{1}{t}}_{L^{\frac{s}{t}}}.$$

But

$$\|I_\alpha |f|^t\|^{\frac{1}{t}}_{L^{\frac{s}{t}}} = \left(\int\limits_{\mathbb{R}^n} \left(\int\limits_{\mathbb{R}^n} \frac{|f(y)|^t}{|x-y|^{n-\alpha}}\, dy \right)^{\frac{s}{t}} dx \right)^{\frac{1}{s}} = \|f\|_{\mathfrak{S}^{t,s,n-\alpha}}.$$

Hence, from the inequality (3.1), it follows that, if $f \in L^{pt}$, then $f \in \mathfrak{S}^{t,s,n-\alpha}$. Therefore we can conclude that $L^{pt} \subseteq \mathfrak{S}^{t,s,n-\alpha}$.

Moreover, the inequality (3.1) tells us that the identity map $I : L^{pt} \to \mathfrak{S}^{t,s,n-\alpha}$ is bounded. ∎

We move to the existence of embedding between Morrey spaces and Stummel spaces. In order to prove it, we need the following lemma.

Lemma 3.2. *Let $1 \leq t < \infty$ and $1 \leq p < \infty$, $0 < \lambda < n$. Then there exist $C_1 > 0$ such that*

$$\||f|^t\|_{L^{p,\lambda}} = \|f\|^t_{L^{pt,\lambda}}.$$

Proof. Observe that for every $x \in \mathbb{R}^n$ and $r > 0$ we have

$$\left(\frac{1}{r^\lambda} \int\limits_{B(x,r)} |f(y)|^{pt}\, dy \right)^{\frac{1}{p}} = \left[\left(\frac{1}{r^\lambda} \int\limits_{B(x,r)} |f(y)|^{pt}\, dy \right)^{\frac{1}{pt}} \right]^t.$$

Hence

$$\sup_{x\in\mathbb{R}^n,r>0}\left(\frac{1}{r^\lambda}\int\limits_{B(x,r)}|f(y)|^{pt}\,dy\right)^{\frac{1}{p}}=\sup_{x\in\mathbb{R}^n,r>0}\left[\left(\frac{1}{r^\lambda}\int\limits_{B(x,r)}|f(y)|^{pt}\,dy\right)^{\frac{1}{pt}}\right]^t.$$

But

$$\sup_{x\in\mathbb{R}^n,r>0}\left[\left(\frac{1}{r^\lambda}\int\limits_{B(x,r)}|f(y)|^{pt}\,dy\right)^{\frac{1}{pt}}\right]^t=\left[\sup_{x\in\mathbb{R}^n,r>0}\left(\frac{1}{r^\lambda}\int\limits_{B(x,r)}|f(y)|^{pt}\,dy\right)^{\frac{1}{pt}}\right]^t.$$

Therefore, we can conclude that

$$\||f|^t\|_{L^{p,\lambda}}=\|f\|^t_{L^{pt,\lambda}},$$

as desired. ∎

We are now ready to prove the existence of embedding between Morrey spaces to Stummel spaces, which is based on the inequality that was established by Chiarenza and Frasca [2].

Theorem 3.3. *Let* $0<\alpha<n$, $1<p<\frac{n}{\alpha}$, *and* $0\le\lambda<n-\alpha p$, *and let* $\frac{1}{q}=\frac{1}{p}-\frac{\alpha}{n-\lambda}$. *Suppose that* $q=\frac{s}{t}$, *for* $1\le t<s<\infty$. *Then we have* $L^{pt,\lambda}\subseteq\mathfrak{S}^{t,s,n-\alpha}_{\lambda}$. *Moreover, the identity map* $I:L^{pt,\lambda}\to\mathfrak{S}^{t,s,n-\alpha}_{\lambda}$ *is bounded.*

Proof. Let $f\in L^{pt,\lambda}$, which is equivalent to $|f|^t\in L^{p,\lambda}$. From Chiarenza-Frasca's theorem, we get that

$$\|I_\alpha|f|^t\|_{L^{q,\lambda}}\le C_{p,q}\||f|^t\|_{L^{p,\lambda}}.$$

Since

$$\||f|^t\|_{L^{p,\lambda}}=\|f\|^t_{L^{pt,\lambda}},$$

we obtain

$$\|I_\alpha|f|^t\|_{L^{q,\lambda}}\le C_{p,q}\|f\|^t_{L^{p,\lambda}}.$$

It thus follows that

$$\|I_\alpha|f|^t\|^{\frac{1}{t}}_{L^{q,\lambda}}\le C'_{p,q}\|f\|_{L^{pt,\lambda}}. \tag{3.2}$$

Observe that, since $q=\frac{s}{t}$, we have

$$\|I_\alpha|f|^t\|^{\frac{1}{t}}_{L^{q,\lambda}}=\|I_\alpha|f|^t\|^{\frac{1}{t}}_{L^{\frac{s}{t},\lambda}}.$$

But

$$\|I_\alpha|f|^t\|^{\frac{1}{t}}_{L^{\frac{s}{t},\lambda}}=\sup_{z\in\mathbb{R}^n,r>0}\left(\frac{1}{r^\lambda}\int\limits_{B(z,r)}\left(\int\limits_{\mathbb{R}^n}\frac{|f(y)|^t}{|x-y|^{n-\alpha}}\,dy\right)^{\frac{s}{t}}dx\right)^{\frac{1}{s}}=\|f\|_{\mathfrak{S}^{t,s,n-\alpha}_{\lambda}}.$$

Hence, from the inequality (3.2), we find that, if $f\in L^{pt,\lambda}$, then $f\in\mathfrak{S}^{t,s,n-\alpha}_{\lambda}$. Therefore, we can conclude that $L^{pt,\lambda}\subseteq\mathfrak{S}^{t,s,n-\alpha}$.

Moreover, the inequality (3.2) tells us that the identity map $I:L^{pt,\lambda}\to\mathfrak{S}^{t,s,n-\alpha}_{\lambda}$ is bounded. ∎

We finally establish the existence of embedding between generalized Morrey spaces and generalized Stummel spaces, which is the ultimate goal of this paper.

Theorem 3.4. *Suppose that* $\phi : (0, \infty) \to (0, \infty)$ *satisfies (2.1), (2.2), and the inequality* $\phi(r) \leq Cr^{\gamma}$ *for* $-\frac{n}{p} \leq \gamma < -\alpha$ *and* $1 < p < \frac{n}{\alpha}$. *Let* $\psi(r) \equiv \phi(r)^{\frac{p}{q}}$ *where* $q = \frac{\gamma p}{\alpha+\gamma}$. *If* $q = \frac{s}{t}$ *for* $1 \leq t < s < \infty$, *then we have* $L^{pt,\phi} \subseteq \mathfrak{S}^{t,s,n-\alpha}_{\psi}$. *Moreover, the identity map* $I : L^{pt,\phi} \to \mathfrak{S}^{t,s,n-\alpha}_{\psi}$ *is bounded.*

Proof. Let $f \in L^{pt,\phi}$, which is equivalent to $|f|^t \in L^{p,\phi}$. From Gunawan-Eridani's theorem, we get that

$$\|I_\alpha |f|^t\|_{L^{q,\psi}} \leq C_{p,q} \||f|^t\|_{L^{p,\phi}}.$$

Since

$$\||f|^t\|_{L^{p,\phi}} = \|f\|^t_{L^{pt,\phi}},$$

we have

$$\|I_\alpha |f|^t\|_{L^{q,\psi}} \leq C_{p,q} \|f\|^t_{L^{p,\phi}}.$$

Accordingly, we obtain

$$\|I_\alpha |f|^t\|^{\frac{1}{t}}_{L^{q,\psi}} \leq C'_{p,q} \|f\|_{L^{pt,\phi}}. \tag{3.3}$$

Observe that, since $q = \frac{s}{t}$, we find that

$$\|I_\alpha |f|^t\|^{\frac{1}{t}}_{L^{q,\psi}} = \|I_\alpha |f|^t\|^{\frac{1}{t}}_{L^{\frac{s}{t},\psi}}.$$

But

$$\begin{aligned} \|I_\alpha |f|^t\|^{\frac{1}{t}}_{L^{\frac{s}{t},\psi}} &= \sup_{z\in\mathbb{R}^n, r>0} \frac{1}{\psi(r)} \left(\frac{1}{|B(z,r)|} \int_{B(z,r)} \left(\int_{\mathbb{R}^n} \frac{|f(y)|^t}{|x-y|^{n-\alpha}}\, dy \right)^{\frac{s}{t}} dx \right)^{\frac{1}{s}} \\ &= \|f\|_{\mathfrak{S}^{t,s,n-\alpha}_{\psi}}. \end{aligned}$$

Hence, from the inequality (3.3), we see that, if $f \in L^{pt,\phi}$, then $f \in \mathfrak{S}^{t,s,n-\alpha}_{\psi}$. Therefore we can conclude that $L^{pt,\phi} \subseteq \mathfrak{S}^{t,s,n-\alpha}_{\psi}$.

Moreover, the inequality (3.3) tells us that the identity map $I : L^{pt,\phi} \to \mathfrak{S}^{t,s,n-\alpha}_{\psi}$ is bounded. ∎

4. Concluding Remarks

We have been successful in establishing the existence of an embedding from Lebesgue spaces to Stummel spaces, and also from (generalized) Morrey spaces to (generalized) Morrey-Stummel spaces, based on the boundedness of the Riesz potential operator on Lebesgue spaces and (generalized) Morrey spaces. With these results, we know now that Stummel spaces and (generalized) Morrey-Stummel spaces are bigger spaces than Lebesgue and (generalized) Morrey spaces, respectively. The topology on these bigger spaces are determined by the norms defined by (1.1) and (2.3), respectively. It is interesting to study the topological and geometrical properties of these spaces, which will be our next project.

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