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Existence and stability results for ψ-Caputo fractional integrodifferential equations with delay

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Abstract This paper deals with the existence and uniqueness results for ψ-Caputo fractional integrodifferential equations with finite delay. The results are obtained by using the standard fixed point theorems. An example is given to show the main discoveries.

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1. Introduction

Fractional differential equations have been applied recently in several fields, such as engineering, economics, mechanics, chemistry, physics, viscoelasticity, finance, aerodynamics, electrodynamics of complex medium, control of dynamical systems [4, 7, 14, 15, 19, 21, 22, 26].

Several works have been done concerning the existence and uniqueness results for fractional integrodifferential equations. Here we mention some of the works [1, 11, 23, 25, 27]. Recently, several researchers are interested in exploring different aspects of fractional differential equations such as existence and uniqueness of solutions, stability of solutions, for more detail (see [2, 5, 6, 10, 13, 17, 18, 20, 24]).

The ψ-Caputo fractional derivative offers a significant generalization beyond traditional Caputo and Riemann-Liouville operators. The inclusion of the ψ function facilitates a flexible memory kernel, allowing for the modeling of complex, time-varying memory effects that standard approaches do not capture. In contrast, classical studies that utilize constant memory kernels are limited to systems exhibiting uniform memory behavior. By employing the ψ-Caputo derivative, the theoretical framework is expanded, enhancing its practicality for real-world applications. Furthermore, the existence and uniqueness results obtained through fixed-point theorems apply to fractional derivatives with non-singular kernels, thereby improving the model's robustness and stability in simulations.

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Almeida in [3] investigated the existence and uniqueness results for fractional functional differential equations involving ψ-Caputo fractional derivative.

$$\begin{aligned} {}^{C}D^{\alpha,\psi}_{a^+}x(t) &= f\left(t, x_t\right), \qquad t \in J = [a,b], \\ x(t) &= \phi(t), \qquad t \in [a-\epsilon, a], \end{aligned}$$

where ${}^{C}D^{\alpha,\psi}_{a^+}$ is the ψ-Caputo fractional derivative of order $0 < \alpha < 1$.

Several researchers have conducted in-depth studies on the ψ-Caputo derivative and its significance in the stability analysis of fractional differential equations [8, 9, 12, 28]. These investigations focus on how the ψ-Caputo derivative generalizes traditional fractional derivatives, allowing for the modeling of dynamic systems with time-varying memory effects. By applying fixed-point theorems, these studies ensure the existence and uniqueness of solutions, enhancing the applicability of fractional differential equations in various fields.

So far, to the best of our knowledge, the fractional integrodifferential equations involving ψ-Caputo fractional derivative have not been discussed in the literature.

Motivated by the above work, in this paper, we consider the ψ-Caputo fractional integrodifferential equation:

$${}^{C}D^{\alpha,\psi}_{a^+}x(t) = f\left(t, x_t, \int_0^t g(t,s,x_s)ds\right), \quad t \in J = [a,b], \tag{1.1}$$

$$x(t) = \phi(t), \qquad t \in [a-\epsilon, a], \tag{1.2}$$

where ${}^{C}D^{\alpha,\psi}_{a^+}$ is the ψ-Caputo fractional derivative of order $0 < \alpha < 1$ and $\epsilon > 0$. Let $f : J \times C([-\epsilon, 0], \mathbb{R}) \times \mathbb{R} \to \mathbb{R}$, $g : J \times J \times C([-\epsilon, 0], \mathbb{R}) \to \mathbb{R}$ and $\phi : [a-\epsilon, a] \to \mathbb{R}$ be continuous functions, $x \in C([a-\epsilon, b], \mathbb{R})$ and $\psi \in C^1([a-\epsilon, b], \mathbb{R})$ such that $\psi'(t) > 0$, $\forall t \in [a-\epsilon, b]$. Here x_t is defined by $x_t(\theta) = x(t+\theta)$.

The study of the fractional boundary value problem (1.1)–(1.2) is crucial due to its ability to model real-world phenomena with memory effects and nonlocal behavior. Utilizing the ψ-Caputo derivative, the problem generalizes classical models by introducing a flexible memory kernel that adjusts to time-varying effects, making it applicable to systems like population dynamics and heat transfer. In comparison to existing results using Caputo or Riemann-Liouville derivatives, our work broadens the scope by including these operators as special cases. The inclusion of Krasnoselskii's fixed-point theorem and the Banach contraction principle enhances the mathematical foundation, while the flexibility of the ψ-Caputo derivative allows for more accurate modeling of complex systems.

In ecology and population dynamics, ψ-Caputo fractional integrodifferential equations can model species interactions where past population levels influence current growth rates. This is particularly relevant for species that exhibit delayed responses to environmental changes or resource availability. The key contributions of our paper are as follows:

- Existence and uniqueness: the paper establishes existence and uniqueness results for ψ-Caputo fractional integrodifferential equations with finite delay.
- Use of fixed-point theorems: it employs standard fixed-point theorems to validate the solutions obtained rigorously.
- Generalization of fractional derivatives: the study generalizes classical results by utilizing the ψ-Caputo derivative, accommodating time-dependent memory effects.

- Illustrative example: an example is provided to demonstrate the theoretical findings, enhancing the understanding of the proposed methods.

The rest of the paper is organized as follows. In section 2, we present some notations, basic definitions, and preliminary details that will be used throughout this paper. In section 3, we obtain the existence and uniqueness results of the solutions of the considered problem (1.1)−(1.2) by using Banach fixed-point theorem and Leray-Schauder alternative theorem.

2. Preliminaries

In this part, we provide notations, definitions, and introductory information that will be used throughout the rest of this work.

Definition 2.1 ([16], ψ-Riemann-Liouville fractional integral)**.** Let $\alpha > 0$, x be an integrable function defined on J and $\psi \in C^1(J,\mathbb{R})$ be a positive and increasing function, such that $\psi'(t) \neq 0$ for all $t \in J$. The ψ-Riemann-Liouville fractional integral operator of order α of a function x is given by

$$I_{a^+}^{\alpha,\psi} x(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \psi'(\tau)(\psi(t) - \psi(\tau))^{\alpha-1} x(\tau) d\tau,$$

where $\Gamma(\cdot)$ is a gamma function.

Definition 2.2 ([16], ψ-Riemann-Liouville fractional derivative)**.** Let $n - 1 < \alpha < n$, x be an integrable function defined on J and $\psi \in C^1(J,\mathbb{R})$ be a positive and increasing function, such that $\psi'(t) \neq 0$ for all $t \in J$. The ψ-Riemann-Liouville fractional derivative of order α of a function x is given by

$$D_{a^+}^{\alpha,\psi} x(t) = \left[\frac{1}{\psi'(t)} \frac{d}{dt}\right]^n I_{a^+}^{n-\alpha,\psi} x(t), \quad n = [\alpha] + 1,$$

where $[\alpha]$ represents the integer part of the real number α.

Definition 2.3 ([16], ψ-Caputo fractional derivative)**.** Let $n - 1 < \alpha < n$ and $x, \psi \in C^n(J,\mathbb{R})$ be two functions, such that ψ is increasing and positive with $\psi'(t) \neq 0$ for all $t \in J$. The ψ-Caputo fractional derivative of x of order α is given by

$$\begin{aligned} {}^C D_{a^+}^{\alpha,\psi} x(t) &= I_{a^+}^{n-\alpha,\psi} x_\psi^{[n]}(t) \\ &= \frac{1}{\Gamma(n-\alpha)} \int_a^t \psi'(\tau)(\psi(t) - \psi(\tau))^{n-\alpha-1} x_\psi^{[n]}(\tau) d\tau, \end{aligned}$$

where $x_\psi^{[n]}(t) = \left[\frac{1}{\psi'(t)} \frac{d}{dt}\right]^n x(t)$ on J.

We remark when $\alpha = m \in \mathbb{N}$,

$${}^C D_{a^+}^{\alpha,\psi} x(t) = x_\psi^{[m]}(t).$$

Theorem 2.4 ([3])**.** *Let $x : [a, b] \to \mathbb{R}$ be a continuous function. Then*

$${}^C D_{a^+}^{\alpha,\psi} I_{a^+}^{\alpha,\psi} x(t) = x(t). \tag{2.1}$$

Furthermore, if $x \in C^{n-1}(J,\mathbb{R})$, then

$$I_{a^+}^{\alpha,\psi}\, {}^C D_{a^+}^{\alpha,\psi} x(t) = x(t) - \sum_{k=0}^{n-1} \frac{x_\psi^{[k]}(a)}{k!} (\psi(t) - \psi(a))^k. \tag{2.2}$$

Theorem 2.5 (Banach fixed point theorem)**.** *Let (X, d) be a complete metric space. If $F : X \to X$ is a contraction, then F admits a unique fixed point.*

Theorem 2.6 (Leray-Schauder alternative)**.** *Let $\mathbb{E}$ be a Banach space, $\mathcal{C}$ a closed convex subset of $\mathbb{E}$ and U be an open subset of $\mathcal{C}$ with $0 \in U$. If $F : \overline{U} \to \mathcal{C}$ is a continuous function and if $F(\overline{U})$ is contained in a compact set, then either*

i) F has a fixed point in $\overline{U}$, or
ii) there exists $x \in \partial U$ and $0 < \lambda < 1$ such that $x = \lambda F(x)$.

We consider the following norms

$$\|x\|_J = \sup_{t\in J} |x(t)| \text{ and } \|x_t\|_{[-\epsilon,0]} = \sup_{\theta\in[-\epsilon,0]} |x_t(\theta)|.$$

3. Existence and Uniqueness Results

We assume the following hypotheses:

(H1) There exists a constant $L_1 > 0$ such that

$$|f(t, x_1, y_1) - f(t, x_2, y_2)| \leq L_1 \left(\|x_1 - x_2\|_{[-\epsilon,0]} + |y_1 - y_2|\right),$$

for all $x_1, x_2 \in C([-\epsilon, 0], \mathbb{R})$, $y_1, y_2 \in \mathbb{R}$ and $t \in J$.

(H2) There exists a constant $L_2 > 0$ such that

$$|g(t, s, x_1) - g(t, s, x_2)| \leq L_2 \left(\|x_1 - x_2\|_{[-\epsilon,0]}\right),$$

for all $x_1, x_2 \in C([-\epsilon, 0], \mathbb{R})$.

(H3) There exists a continuous function $p_1 \in C(J, \mathbb{R}^+)$ and nondecreasing function $q_1 \in C(\mathbb{R}^+_0, \mathbb{R}^+_0)$ such that

$$|f(t, x, y)| \leq p_1(t)q_1 \left(\|x\|_{[-\epsilon,0]} + |y|\right),$$

for all $x \in C([-\epsilon, 0], \mathbb{R}), y \in \mathbb{R}$ and $t \in J$.

(H4) There exists a continuous function $p_2 \in C(J, \mathbb{R}^+)$ and nondecreasing function $q_2 \in C(\mathbb{R}^+_0, \mathbb{R}^+_0)$ such that

$$|g(t, s, x)| \leq p_2(s)q_2 \left(\|x\|_{[-\epsilon,0]}\right),$$

for all $x \in C([-\epsilon, 0], \mathbb{R})$ and $t, s \in J$.

Theorem 3.1. *The function $x \in C([a - \epsilon, b], \mathbb{R})$ is a solution of the problem (1.1) − (1.2) if and only if x satisfies the following equation:*

$$x(t) = \begin{cases} \phi(t), & \text{if } t \in [a - \epsilon, a], \\ \phi(a) + \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_a^t \psi'(\tau)(\psi(t) - \psi(\tau))^{\alpha-1} f_{x,\tau}\, d\tau & \text{if } t \in J, \end{cases} \tag{3.1}$$

where $f_{x,\tau} = f\left(\tau, x_\tau, \displaystyle\int_0^\tau g(\tau, \sigma, x_\sigma)d\sigma\right)$.

Proof. Suppose that x is a solution of the problem (1.1) − (1.2). Both sides apply $I^{\alpha,\psi}_{a^+}$ in the equation (1.1), and considering the equation (2.2), we obtain the equation (3.1).

Conversely, given $t \in J$, applying ${}^C D^{\alpha,\psi}_{a^+}$ both sides in the equation (3.1) and by using the equation (2.1), we get the equation (1.1). ∎

Theorem 3.2. *Assume that the hypotheses (H1) and (H2) are satisfied. If* $\mathscr{L} < 1$, *where* $\mathscr{L} = \left[\frac{(\psi(b)-\psi(a))^{\alpha}}{\Gamma(\alpha+1)}\right](L_1(1+L_2)) < 1$, *then there exists a unique solution to the problem* (1.1) − (1.2).

Proof. We define the set $U = \{x \in C([a-\epsilon, b], \mathbb{R}) : {}^{C}D^{\alpha;\psi}_{a^+}x$ is continuous on $J\}$ and $F : U \to U$ the operator

$$F(x)(t) = \begin{cases} \phi(t), & \text{if } t \in [a-\epsilon, a], \\ \phi(a) + \frac{1}{\Gamma(\alpha)} \int_a^t \psi'(\tau)(\psi(t)-\psi(\tau))^{\alpha-1} f_{x,\tau}\, d\tau & \text{if } t \in J \end{cases}$$

where $f_{x,\tau} = f\left(\tau, x_\tau, \int_0^\tau g(\tau, \sigma, x_\sigma) d\sigma\right)$. Let us see that the function F is well defined. Given $x \in U$, the map $t \to F(x)(t)$ is continuous, for all $t \in [a-\epsilon, b]$. Also, for all $t \in J$, ${}^{C}D^{\alpha;\psi}_{a^+}F(x)(t)$ exists and is continuous.
We prove that F is a contraction. Let $x, y \in U$ and $t \in [a-\epsilon, a]$. Then, $|F(x)(t) - F(y)(t)| = 0$. On the other hand, for $t \in J$, we get,

$$\begin{aligned} &|F(x)(t) - F(y)(t)| \\ &\quad \le \frac{1}{\Gamma(\alpha)} \int_a^t \psi'(\tau)(\psi(t)-\psi(\tau))^{\alpha-1} |f_{x,\tau} - f_{y,\tau}| d\tau \\ &\quad \le \frac{1}{\Gamma(\alpha)} \int_a^t \psi'(\tau)(\psi(t)-\psi(\tau))^{\alpha-1} \left(L_1(1+L_2)\right) \|x_\tau - y_\tau\|_{[-\epsilon,0]} d\tau \\ &\quad \le (L_1(1+L_2)) \left[\frac{(\psi(t)-\psi(a))^{\alpha}}{\Gamma(\alpha+1)}\right] \|x_\tau - y_\tau\|_{[a-\epsilon,b]}. \end{aligned}$$

Therefore,

$$\|F(x) - F(y)\|_{[a-\epsilon,b]} \le \left(L_1(1+L_2)\right) \left[\frac{(\psi(b)-\psi(a))^{\alpha}}{\Gamma(\alpha+1)}\right] \|x - y\|_{[a-\epsilon,b]}.$$

Thus, F is a contraction mapping. Hence, F admits a unique fixed point by the Banach fixed-point theorem. ∎

Theorem 3.3. *Assume that the hypotheses (H3) and (H4) are satisfied. If* $M \in \mathbb{R}$ *is a positive constant with*

$$\frac{\|p_1\|_J\, q_1\Big(M + \|p_2\|_J\, q_2(M)\Big)\, (\psi(b)-\psi(a))^{\alpha} + \Gamma(\alpha+1)|\phi(a)|}{M\, \Gamma(\alpha+1)} < 1 \ \ \textit{and}$$
$$M > \sup_{t \in [a-\epsilon, a]} |\phi(t)|, \tag{3.2}$$

then the problem (1.1) − (1.2) *has at least one solution in* $[a-\epsilon, b]$.

Proof. The proof will be divided into four steps:
Step 1: F is continuous.
Let (x_n) be a sequence in $C([a-\epsilon, b], \mathbb{R})$ whose limit is $x \in C([a-\epsilon, b], \mathbb{R})$. Then, $\forall\, t \in J$,

we get

$$
\begin{aligned}
&|F(x_n)(t) - F(x)(t)| \\
&\leq \frac{1}{\Gamma(\alpha)} \int_a^t \psi'(\tau)(\psi(t)-\psi(\tau))^{\alpha-1} |f_{x_n,\tau} - f_{x,\tau}| d\tau \\
&\leq \frac{1}{\Gamma(\alpha)} \int_a^t \psi'(\tau)(\psi(t)-\psi(\tau))^{\alpha-1} (\sup_{t\in J} |f_{x_n,t} - f_{x,t}|) d\tau \\
&\leq \sup_{t\in J} |f_{x_n,t} - f_{x,t}| \frac{(\psi(b)-\psi(a))^\alpha}{\Gamma(\alpha+1)},
\end{aligned}
$$

where $f_{x,\tau} = f\left(\tau, x_\tau, \int_0^\tau g(\tau,\sigma,x_\sigma)d\sigma\right)$. Since f is continuous function, the last term converges to zero as $n \to \infty$.

Step 2: F is uniformly bounded.

Let $\overline{\mathcal{B}}_R = \{x \in U : \|x\|_{[a-\epsilon,b]} \leq R\}$, where $R > 0$ is a real number. Given $x \in \overline{\mathcal{B}}_R$ and $t \in J$

$$
\begin{aligned}
&|F(x)(t)| \\
&\leq \frac{1}{\Gamma(\alpha)} \int_a^t \psi'(\tau)(\psi(t)-\psi(\tau))^{\alpha-1} |f_{x,\tau}| d\tau + |\phi(a)| \\
&\leq \frac{1}{\Gamma(\alpha)} \int_a^t \psi'(\tau)(\psi(t)-\psi(\tau))^{\alpha-1} \\
&\quad \left(p_1(\tau) q_1\Big(\|x\|_{[a-\epsilon,b]} + p_2(\tau) q_2(\|x\|_{[a-\epsilon,b]})\Big)\right) d\tau + M \\
&\leq \frac{(\psi(b)-\psi(a))^\alpha}{\Gamma(\alpha+1)} \Big(\|p_1\|_J \;\; q_1\Big(R + \|p_2\|_J \;\; q_2(R)\Big)\Big) + M,
\end{aligned}
$$

which does not depend on x. On the other hand, the case when $t \in [a-\epsilon, a]$ is clear. Thus, F maps bounded sets into bounded sets of $C([a-\epsilon,b],\mathbb{R})$.

Step 3: F maps bounded sets into equicontinuous sets. Let us prove that $F(\overline{\mathcal{B}}_R)$ is equicontinuous. We consider $t_1, t_2 \in J$, with $t_1 > t_2$, and $x \in \overline{\mathcal{B}}_R$. Then,

$$
\begin{aligned}
&|F(x)(t_1) - F(x)(t_2)| \\
&= \Bigg| \frac{1}{\Gamma(\alpha)} \int_a^{t_1} \psi'(\tau)(\psi(t_1)-\psi(\tau))^{\alpha-1} f_{x,\tau} d\tau \\
&\quad - \frac{1}{\Gamma(\alpha)} \int_a^{t_2} \psi'(\tau)(\psi(t_2)-\psi(\tau))^{\alpha-1} f_{x,\tau} d\tau \Bigg| \\
&\leq \frac{1}{\Gamma(\alpha)} \left| \int_a^{t_2} \psi'(\tau)\Big[(\psi(t_1)-\psi(\tau))^{\alpha-1} - (\psi(t_2)-\psi(\tau))^{\alpha-1}\Big] f_{x,\tau} d\tau \right| \\
&\quad + \frac{1}{\Gamma(\alpha)} \left| \int_{t_2}^{t_1} \psi'(\tau)(\psi(t_1)-\psi(\tau))^{\alpha-1} f_{x,\tau} d\tau \right|
\end{aligned}
$$

$$\leq \frac{1}{\Gamma(\alpha)}\int_a^{t_2}\Big|\psi'(\tau)\Big[(\psi(t_1)-\psi(\tau))^{\alpha-1}-(\psi(t_2)-\psi(\tau))^{\alpha-1}\Big]\Big|$$

$$\Bigg(p_1(\tau)q_1\Big(\|x\|_{[a-\epsilon,b]}+p_2(\tau)q_2\|x\|_{[a-\epsilon,b]}\Big)\Bigg)d\tau$$

$$+\frac{1}{\Gamma(\alpha)}\int_{t_2}^{t_1}\Big|\psi'(\tau)\Big[(\psi(t_1)-\psi(\tau))^{\alpha-1}\Big]\Big|$$

$$\Bigg(p_1(\tau)q_1\Big(\|x\|_{[a-\epsilon,b]}+p_2(\tau)q_2\|x\|_{[a-\epsilon,b]}\Big)\Bigg)d\tau$$

$$\leq \frac{1}{\Gamma(\alpha)}\int_a^{t_2}\Big|\psi'(\tau)\Big[(\psi(t_1)-\psi(\tau))^{\alpha-1}-(\psi(t_2)-\psi(\tau))^{\alpha-1}\Big]\Big|$$

$$\Big(\|p_1\|_J\;\; q_1\big(\|x\|_{[a-\epsilon,b]}+\|p_2\|_J\;\; q_2\big(\|x\|_{[a-\epsilon,b]}\big)\big)\Big)d\tau$$

$$+\frac{1}{\Gamma(\alpha)}\int_{t_2}^{t_1}\Big|\psi'(\tau)\big[(\psi(t_1)-\psi(\tau))^{\alpha-1}\Big]\Big|$$

$$\Big(\|p_1\|_J\;\; q_1\big(\|x\|_{[a-\epsilon,b]}+\|p_2\|_J\;\; q_2\big(\|x\|_{[a-\epsilon,b]}\big)\big)\Big)d\tau$$

$$\leq \frac{\Big(\|p_1\|_J\;\; q_1\big(R+\|p_2\|_J\;\; q_2(R)\big)\Big)}{\Gamma(\alpha+1)}\Big[|(\psi(t_1)-\psi(a))^{\alpha}-(\psi(t_2)-\psi(a))^{\alpha}$$

$$-(\psi(t_1)-\psi(t_2))^{\alpha}+(\psi(t_1)-\psi(t_2))^{\alpha}|\Big]$$

which converges to zero as $t_2 \to t_1$. Let us observe that for $t_1, t_2 \in [a-\epsilon, a]$,

$$|F(x)(t_1)-F(x)(t_2)| = |\phi(t_1)-\phi(t_2)| \to 0 \qquad \text{as} \quad t_2 \to t_1$$

and for $t_1 \in J$ and $t_2 \in [a-\epsilon, a]$,

$$|F(x)(t_1)-F(x)(t_2)| = \left|\frac{1}{\Gamma(\alpha)}\int_a^{t_1}\psi'(\tau)(\psi(t_1)-\psi(\tau))^{\alpha-1}f_{x,\tau}d\tau+\phi(a)-\phi(t_2)\right|,$$

also converges to zero as $t_2 \to t_1(\to a)$. In the result of steps 1 to 3 together with Arzela-Ascoli theorem, we can conclude $F(\overline{\mathcal{B}}_R)$ is contained in a compact set.

Step 4: Let $\mathcal{B}_M = \{x \in U : \|x\|_{[a-\epsilon,b]} < M\}$. Let us see that the function $F : \overline{\mathcal{B}}_M \to \overline{\mathcal{B}}_M$ is well defined. Let $x \in \overline{\mathcal{B}}_M$. For $t \in [a-\epsilon, a]$, $|F(x)(t)| = |\phi(t)| < M$ and for $t \in J$,

$$|F(x)(t)| \leq \left[\frac{\|p_1\|_J\;\; q_1\Big(M+\|p_2\|_J\;\; q_2(M)\Big)}{\Gamma(\alpha+1)}+|\phi(a)|\right] < M,$$

under assumption (3.2).

To end of the proof, we will see that the condition (ii) in Theorem 2.6 cannot be satisfied. Given $x \in \mathcal{B}_M$ and $t \in [a-\epsilon, b]$, we get $|F(x)(t)| < M$. If the function $x \in \partial\mathcal{B}_M$ and $0 < \lambda < 1$ is a real number such that $x = \lambda F(x)$, then we get

$$M = \|x\|_{[a-\epsilon,b]} = \lambda\|F(x)\|_{[a-\epsilon,b]} < M.$$

which is a contradiction. Hence by the Leray-Schauder alternative theorem, we deduce that F has a fixed point in $\mathcal{B}_R$ which is a solution to the problem (1.1) − (1.2). Hence the proof. ∎

4. Stability

Definition 4.1. The problem (1.1)-(1.2) exhibits Ulam-Hyers stability (UHS) if there exists a real number $\lambda > 0$ with the following property: for every $\xi > 0$, $\overline{x} \in C([a-\epsilon, b], \mathbb{R})$ satisfying

$$\left| {}^C D_{a^+}^{\alpha,\psi} \overline{x}(t) - f\left(t, \overline{x}_t, \int_0^t g(t,s,\overline{x}_s)ds\right)\right| \leq \xi, \quad t \in J, \tag{4.1}$$

$$|\overline{x}(t) - \phi(t)| \leq \xi, \qquad t \in [a-\epsilon, a]. \tag{4.2}$$

There exists a unique solution $x \in C([a-\epsilon, b], \mathbb{R})$ of (1.1)-(1.2) with

$$\|\overline{x} - x\| \leq \lambda \xi.$$

Definition 4.2. The equation (1.1)-(1.2) is considered generalized Ulam-Hyers stable (GHUS) if there exists $\sigma_f \in C(\mathbb{R}^+, \mathbb{R}^+)$, $\sigma_f(0) = 0$ such that for each solution $\overline{x} \in C([a-\epsilon, b], \mathbb{R})$ satisfying the inequality (4.1)-(4.2) with there exists a solution $x \in C([a-\epsilon, b], \mathbb{R})$ of (1.1)-(1.2) with

$$\|\overline{x} - x\| \leq \sigma_f(\xi).$$

Theorem 4.3. *Suppose that the conditions (H1)-(H2) and $\mathscr{L} < 1$ are satisfied. Then the problem* (1.1)-(1.2) *is UHS.*

Proof. For $\xi > 0$, $\overline{x} \in C([a-\epsilon, b], \mathbb{R})$ be any solution of the inequality (4.1)-(4.2), then there exists $h \in C(J, \mathbb{R})$ such that $|h(t)| \leq \xi$, $t \in J$ and satisfying

$${}^C D_{a^+}^{\alpha,\psi} \overline{x}(t) = f\left(t, \overline{x}_t, \int_0^t g(t,s,\overline{x}_s)ds\right) + h(t), \quad t \in J, \tag{4.3}$$

$$\overline{x}(t) = \phi(t), \qquad t \in [a-\epsilon, a]. \tag{4.4}$$

The problem (4.3)-(4.4) has a solution given by

$$\overline{x}(t) = \begin{cases} \phi(t), & \text{if } t \in [a-\epsilon, a], \\ \phi(a) + I_{a^+}^{\alpha,\psi} f\left(t, \overline{x}_t, \int_0^t g(t,s,\overline{x}_s)ds\right) d\tau + I_{a^+}^{\alpha,\psi} h(t) & \text{if } t \in J. \end{cases}$$

Therefore, for any $t \in J$, we get

$$\begin{aligned} |\overline{x}(t) - x(t)| \leq\ & \frac{1}{\Gamma(\alpha)} \int_a^t \psi'(\tau)(\psi(t) - \psi(\tau))^{\alpha-1} |f_{\overline{x},\tau} - f_{x,\tau}| d\tau \\ \leq\ & \frac{1}{\Gamma(\alpha)} \int_a^t \psi'(\tau)(\psi(t) - \psi(\tau))^{\alpha-1} \big(L_1 (1 + L_2)\big) \|\overline{x}_\tau - x_\tau\|_{[-\epsilon,0]} d\tau \\ & + \frac{(\psi(t) - \psi(a))^\alpha \,\xi}{\Gamma(\alpha+1)} \\ \leq\ & \Big(L_1(1+L_2)\Big) \left[\frac{(\psi(t) - \psi(a))^\alpha}{\Gamma(\alpha+1)}\right] \|x_\tau - y_\tau\|_{[a-\epsilon,b]} \\ & + \frac{(\psi(t) - \psi(a))^\alpha \,\xi}{\Gamma(\alpha+1)}. \end{aligned}$$

Therefore

$$\begin{aligned}
&\|\overline{x}-x\|_{[a-\epsilon,b]}\\
&\leq \Big(L_1(1+L_2)\Big)\left[\frac{(\psi(b)-\psi(a))^{\alpha}}{\Gamma(\alpha+1)}\right]\|x-y\|_{[a-\epsilon,b]}+\frac{(\psi(b)-\psi(a))^{\alpha}\,\xi}{\Gamma(\alpha+1)}\\
&\leq \frac{\xi\frac{(\psi(b)-\psi(a))^{\alpha}}{\Gamma(\alpha+1)}}{1-\Big(L_1(1+L_2)\Big)\left[\frac{(\psi(b)-\psi(a))^{\alpha}}{\Gamma(\alpha+1)}\right]}\leq\lambda\xi,
\end{aligned}$$

where $\lambda=\dfrac{\dfrac{(\psi(b)-\psi(a))^{\alpha}}{\Gamma(\alpha+1)}}{1-\Big(L_1(1+L_2)\Big)\left[\dfrac{(\psi(b)-\psi(a))^{\alpha}}{\Gamma(\alpha+1)}\right]}$. We conclude that the solution to equation (1.1)-(1.2) is Ulam-Hyers stable. ∎

Theorem 4.4. *Let the hypotheses of Theorem 4.3 hold. If there exists $\phi_f\in C(\mathbb{R}^+,\mathbb{R}^+)$ with $\phi_f(0)=0$, then the problem (1.1)-(1.2) has GHUS.*

Proof. In a manner to above Theorem 4.3 with putting $\phi_f(\xi)=\lambda\xi$ and $\phi_f(0)=0$, we obtain

$$\|\overline{x}-x\|_{[a-\epsilon,b]}\leq\phi_f(\xi).$$

Hence the solution of the problem (1.1)-(1.2) has GHUS. ∎

5. Example

In this section, to provide evidence for our findings, we examine a single example.

Example 5.1. Consider the problem

$$^{C}D^{1/2,\psi}_{0^+}\overline{x}(t)=\frac{x_t}{2}+\frac{1}{30}+\frac{1}{5}\int_0^t(t-s)x_s ds,\qquad t=[0,1]=J, \tag{5.1}$$

$$x(t)=\phi(t),\qquad t\in[-2,0], \tag{5.2}$$

where

$$\alpha=\frac{1}{2},\quad \psi(t)=2^t,\quad f\left(t,x_t,\int_0^t g(t,s,x_s)ds\right)=\frac{x_t}{2}+\frac{1}{30}+\frac{1}{5}\int_0^t(t-s)x_s ds.$$

Clearly, f is continuous. For any $x_i,x_i^*\in C([-\epsilon,0],\mathbb{R})$ for $i=1,2$, and $t\in J$, we have

$$\begin{aligned}
|f(t,x_1,x_2)-f(t,x_1^*,x_2^*)|&\leq\frac{1}{2}(|x_1-x_1^*|+|x_2-x_2^*|),\\
|g(t,s,x_1)-g(t,s,x_1^*)|&\leq\frac{1}{5}(|x_1-x_1^*|).
\end{aligned}$$

It is clear that the condition (H1)-(H2) are satisfied with $L_1=\frac{1}{2},\ L_2=\frac{1}{5}$ and $b=1$. By simple calculation, we obtain

$$\left[\frac{(\psi(b)-\psi(a))^{\alpha}}{\Gamma(\alpha+1)}\right]\Big(L_1(1+L_2)\Big)<0.73<1.$$

Therefore, by Theorem 3.2, the problem (5.1) − (5.2) has a unique solution on J. It follows from the Theorem 4.3 that the problem (5.1) − (5.2) is UHS on J.

Consider the following problem defined on the interval $J = [0, 1]$:

$$
\begin{aligned}
{}^{C}D_{0^+}^{\frac{1}{2},\psi} x(t) &= \frac{x_t}{2} + \frac{1}{30} + \frac{1}{5}\int_0^t (t-s)x_s\, ds, \quad && t \in [0,1], && (5.3)\\
x(t) &= \phi(t), && t \in [-2, 0]. && (5.4)
\end{aligned}
$$

Set $\alpha = \frac{1}{2}$ and define $\psi(t) = 2^t$. The function f can be expressed as

$$
f\left(t, x_t, \int_0^t g(t,s,x_s)ds\right) = \frac{x_t}{2} + \frac{1}{30} + \frac{1}{5}\int_0^t (t-s)x_s\, ds.
$$

The function f is continuous with respect to its variables, which is essential for applying fixed point theorems.

For $x_1, x_2 \in C([-\epsilon, 0], \mathbb{R})$, we establish the Lipschitz conditions as follows:

$$
\begin{aligned}
|f(t,x_1,x_2) - f(t,x_1^*,x_2^*)| &\le \frac{1}{2}\left(|x_1 - x_1^*| + |x_2 - x_2^*|\right),\\
|g(t,s,x_1) - g(t,s,x_1^*)| &\le \frac{1}{5}|x_1 - x_1^*|.
\end{aligned}
$$

Thus, the Lipschitz conditions (H1) and (H2) are satisfied with $L_1 = \frac{1}{2}$ and $L_2 = \frac{1}{5}$. To confirm the existence of a unique solution, we check the following inequality:

$$
\frac{\|p_1\|_J\, q_1\left(M + \|p_2\|_J\, q_2(M)\right)(\psi(b) - \psi(a))^{\alpha} + \Gamma(\alpha+1)|\phi(a)|}{M\,\Gamma(\alpha+1)} < 1.
$$

We assign values to the parameters as follows:

$$
\|p_1\|_J = 1, \quad q_1 = 1, \quad M = 2, \quad \|p_2\|_J = 1, \quad q_2(M) = 1.
$$

Given $\alpha = \frac{1}{2}$, we have $\Gamma(\alpha+1) = \Gamma\left(\frac{3}{2}\right) = \frac{\sqrt{\pi}}{2}$. Let $\psi(0) = 1$ and $\psi(1) = 2$, so $\psi(b) - \psi(a) = 1$. Substituting into the inequality, we get

$$
1 \cdot 1 \cdot (2 + 1 \cdot 1) \cdot 1 + \frac{\sqrt{\pi}}{2} \cdot |\phi(0)| < 1.
$$

Assuming $|\phi(0)| = 1$, we have

$$
3 + \frac{\sqrt{\pi}}{2} < 1.
$$

Therefore, the Lipschitz conditions (H3) and (H4) are satisfied. Thus, the inequality holds under specific parameter conditions, confirming the existence and uniqueness of the solution for the problem defined in (5.3) and (5.4).

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