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An equivalent norm of Herz spaces and its application to the Carleson operator

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Abstract By establishing a new norm equivalence on Herz spaces using the Muckenhoupt class, the boundedness of the maximal modulated singular integral operator is established. This boundedness also boils down to the boundedness of the Carleson operator over the real line.

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1. Introduction

The goal of this note is twofold. One is to obtain an equivalent norm to the (homogeneous) Herz norm $\|\cdot\|_{\dot{K}_p^{\alpha,q}}$ by means of weights for the parameters $1 < p < \infty$, $0 < q \le \infty$, and $\alpha \in \mathbb{R}$ and the other is to obtain the boundedness of the Carleson operator as an application of this norm equivalence.

For each $k \in \mathbb{Z}$, we set $C_k \equiv [-2^k, 2^k]^n \setminus [-2^{k-1}, 2^{k-1}]^n$ and denote by χ_k its indicator function. Then the homogeneous Herz norm $\|\cdot\|_{\dot{K}_p^{\alpha,q}}$ is given by

$$\|f\|_{\dot{K}_p^{\alpha,q}} \equiv \left(\sum_{j=-\infty}^{\infty} (2^{j\alpha} \|f\chi_j\|_{L^p})^q \right)^{\frac{1}{q}}$$

for a measurable function f. The homogeneous Herz space $\dot{K}_p^{\alpha,q}(\mathbb{R}^n)$ is the set of all measurable functions f for which $\|f\|_{\dot{K}_p^{\alpha,q}}$ is finite.

By a weight we mean a non-negative measurable function which is almost everywhere positive and finite. For a weight w and $1 < p < \infty$, we define the weighted L^p-norm by $\|f\|_{L^p(w)} \equiv \|w^{\frac{1}{p}} f\|_{L^p}$ for a measurable function f. We seek to establish the following norm equivalence to obtain the boundedness of the Carleson operator:

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Theorem 1.1. *Let $1 < p < \infty$, $0 < q \le \infty$, and $\alpha \in \mathbb{R}$. Suppose that we have parameters $\kappa_1, \kappa_2 \in \mathbb{R}$ satisfying*

$$\frac{\kappa_1}{p} < \alpha < \frac{\kappa_2}{p}. \tag{1.1}$$

Define

$$w(x) \equiv \min(|x|^{\kappa_1}, |x|^{\kappa_2}) \tag{1.2}$$

for $x \in \mathbb{R}^n$ and set

$$w_l \equiv w(2^{-l}\cdot) \quad (l \in \mathbb{Z}). \tag{1.3}$$

Then we have an equivalence of norms:

$$\|f\|_{\dot{K}^{\alpha,q}_p} \sim \left(\sum_{l=-\infty}^{\infty} (2^{l\alpha}\|f\|_{L^p(w_l)})^q\right)^{\frac{1}{q}}$$

for a measurable function $f : \mathbb{R}^n \to \mathbb{C}$, where the implicit constants in $\sim$ are independent of f.

As an application, we will establish the boundedness of maximally modulated singular integrals. To this end, we take a smooth function $m \in C^\infty(\mathbb{R}^n \setminus \{0\})$, called the Fourier multiplier, satisfying

$$|\partial^\alpha m(\xi)| \lesssim |\xi|^{-|\alpha|} \quad (\xi \in \mathbb{R}^n \setminus \{0\}) \tag{1.4}$$

for all multiindices α. We define the maximally modulated singular integral operator $\mathcal{C}_m$ with the Fourier multiplier m by

$$\mathcal{C}_m f(x) \equiv \sup_{\xi \in \mathbb{R}^n} |\mathcal{F}^{-1}[m(\cdot - \xi)\mathcal{F}f](x)| \quad (x \in \mathbb{R}^n)$$

for $f \in L^2(\mathbb{R}^n)$. It is known that $\mathcal{C}_m$, initially defined for functions in $L^2(\mathbb{R}^n)$ and satisfies $|\mathcal{C}_m f - \mathcal{C}_m g| \le \mathcal{C}_m[f - g]$ for any $f, g \in L^2(\mathbb{R}^n)$, extends to a bounded sublinear operator on $L^p(\mathbb{R}^n)$ for all $1 < p < \infty$. See [7, Theorem 1.1]. Denote by p' its conjugate exponent for $1 < p < \infty$. Bearing in mind that $L^\infty_c(\mathbb{R}^n)$, the linear space of all essentially bounded functions with compact support, is dense in $\dot{K}^{\alpha,q}_p(\mathbb{R}^n)$, we will prove the following theorem:

Theorem 1.2. *Suppose that $m \in C^\infty(\mathbb{R}^n \setminus \{0\})$ satisfies (1.4). Then the maximally modulated singular integral operator $\mathcal{C}_m$ extends to a bounded linear operator on $\dot{K}^{\alpha,q}_p(\mathbb{R}^n)$ for all $1 < p < \infty$, $0 < q \le \infty$, and all $\alpha \in \mathbb{R}$ with $-\frac{n}{p} < \alpha < \frac{n}{p'}$.*

Remark that the special case where $n = 1$ and $m = \chi_{(0,\infty)} - \chi_{(-\infty,0)}$ covers the Carleson operator $\mathcal{C} = \mathcal{C}_{\chi_{(0,\infty)} - \chi_{(-\infty,0)}}$, which plays the central role of proving the almost everywhere convergence of the L^2-Fourier series and the L^2-Fourier transform. As a result, we have the following conclusion:

Theorem 1.3. *The Carleson operator $\mathcal{C}$ is bounded on $\dot{K}^{\alpha,q}_p(\mathbb{R})$ for all $1 < p < \infty$, $0 < q \le \infty$, and all $\alpha \in \mathbb{R}$ with $-\frac{1}{p} < \alpha < \frac{1}{p'}$.*

As a generalization, we can replace $\mathcal{C}_m$ by higher order commutators given by

$$\mathcal{C}_{m,b,k} f(x) \equiv \sup_{\xi \in \mathbb{R}^n} |\mathcal{F}^{-1}[m(\cdot - \xi)\mathcal{F}[(b(x) - b(\cdot))^k f]](x)| \quad (x \in \mathbb{R}^n),$$

where $b \in \mathrm{BMO}(\mathbb{R}^n)$ and $k \in \mathbb{N}$. If we reexamine the proof of Theorem 1.2 using [9, p. 543, C], we see that we can replace $\mathcal{C}_m$ by $\mathcal{C}_{m,b,k}$. We do not pursue this direction.

We have an analogy to nonhomogeneous Herz spaces. Let $1 < p < \infty$, $0 < q \le \infty$, and $\alpha \in \mathbb{R}$. Then the nonhomogeneous Herz norm $\| \cdot \|_{K_p^{\alpha,q}}$ is given by

$$\|f\|_{K_p^{\alpha,q}} \equiv \|f\chi_{[-1,1]^n}\|_{L^p} + \left(\sum_{j=1}^{\infty}(2^{j\alpha}\|f\chi_j\|_{L^p})^q\right)^{\frac{1}{q}}$$

for a measurable function f. The nonhomogeneous Herz space $K_p^{\alpha,q}(\mathbb{R}^n)$ is the set of all measurable functions f for which $\|f\|_{K_p^{\alpha,q}}$ is finite.

Since the proof goes parallelly to $\dot{K}_p^{\alpha,q}(\mathbb{R}^n)$, we do not formulate or prove counterparts to Theorems 1.1–1.3 except the statement of a counterpart of nonhomogeneous Herz spaces to Theorem 1.1.

Theorem 1.4. *Let $1 < p < \infty$, $0 < q \le \infty$, and $\alpha \in \mathbb{R}$. Suppose that we have parameters $\kappa_1, \kappa_2, \kappa_3 \in \mathbb{R}$ satisfying (1.1) and $\frac{\kappa_3}{p} < \alpha$. Define w and w_l, $l \in \mathbb{N}$ by (1.2) and (1.3), respectively. We define*

$$v(x) \equiv \min(1, |x|^{\kappa_3}) \quad (x \in \mathbb{R}^n). \tag{1.5}$$

Then we have an equivalence of norms:

$$\|f\|_{\dot{K}_p^{\alpha,q}} \sim \|f\|_{L^p(v)} + \left(\sum_{l=1}^{\infty}(2^{l\alpha}\|f\|_{L^p(w_l)})^q\right)^{\frac{1}{q}}$$

for a measurable function $f : \mathbb{R}^n \to \mathbb{C}$, where the implicit constants in $\sim$ are independent of f.

Remark that the technique employed in the present paper is applicable to many operators such as singular integral operators, commutators generated by BMO and singular integral operators as well as the fractional integral operator I_β, where I_β is an operator defined for suitable functions f by

$$I_\beta f(x) \equiv \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\beta}} dy \ (x \in \mathbb{R}^n).$$

We can reprove the following theorem by Li and Yang [4]:

Theorem 1.5. *Let $0 < \beta < n$, $1 < p_1 < p_2 < \infty$, $0 < q \le \infty$ satisfy*

$$-\frac{n}{p_2} < \alpha < \frac{n}{p_1'}, \quad \frac{1}{p_2} = \frac{1}{p_1} - \frac{\beta}{n}. \tag{1.6}$$

Then I_β maps $\dot{K}_{p_1}^{\alpha,q}(\mathbb{R}^n)$ to $\dot{K}_{p_2}^{\alpha,q}(\mathbb{R}^n)$.

See [5] for more about the recent approach on Herz spaces. The technique of proving Theorem 1.2 promises more applications to inequalities there as well as other operators in the existing literature. For example, we can deal with multilinear commutators of singular integrals based on the sharp maximal inequality obtained in [3]. As for the multilinear commutators of fractional integral operators, we can use the estimate in [1]. Further details are omitted.

We use the following standard notation for inequalities: Let $A, B \ge 0$. Then $A \lesssim B$ and $B \gtrsim A$ mean that there exists a constant $C > 0$ such that $A \le CB$, where C depends only on the parameters of importance. The symbol $A \sim B$ means that $A \lesssim B$ and $B \lesssim A$ happen simultaneously.

The remaining part of this paper is organized as follows: Section 2 proves Theorem 1.1. Section 3 proves Theorem 1.2. We end this paper with proving Theorem 1.5 in Section 4.

2. Proof of Theorem 1.1

We suppose $q < \infty$ for the sake of simplicity; otherwise the proof below can be modified readily. Let $f : \mathbb{R}^n \to \mathbb{C}$ be a measurable function. Then the one-sided estimate

$$\|f\|_{\dot{K}_p^{\alpha,q}} \lesssim \left(\sum_{l=-\infty}^{\infty} (2^{l\alpha} \|f\|_{L^p(w_l)})^q \right)^{\frac{1}{q}}$$

is easy to show, since

$$\chi_l \lesssim w_l. \tag{2.1}$$

For the converse estimate, we use

$$\|f\|_{L^p(w_l)} \lesssim \sum_{j=-\infty}^{\infty} \min(2^{\frac{\kappa_1}{p}(l-j)}, 2^{\frac{\kappa_2}{p}(l-j)}) \|f\chi_j\|_{L^p}$$

with the assumption $1 < p < \infty$ in mind. Let $\varepsilon \equiv \min\left(\alpha - \frac{\kappa_1}{p}, \frac{\kappa_2}{p} - \alpha\right) > 0$. If we use the boundedness of the discrete Hardy operator which guarantees

$$\sum_{l=-\infty}^{\infty} \left(\sum_{j=-\infty}^{\infty} 2^{-\varepsilon|l-j|} |a_j| \right)^q \lesssim \sum_{l=-\infty}^{\infty} |a_l|^q$$

for all $\{a_j\}_{j=-\infty}^{\infty} \in \ell^q$ (see [8, Proposition 1.2] for example), then we obtain the reverse inequality.

3. Proof of Theorem 1.2

We use the notion of the Muckenhoupt class A_p. By a "cube" we mean a compact cube whose edges are parallel to the coordinate axes. We denote by $\mathcal{Q}$ the set of all cubes. Let E be a measurable set and f be a measurable function with respect to the Lebesgue measure. Then write $m_E(f) \equiv \frac{1}{|E|} \int_E f(x) \mathrm{d}x$. Let $1 < p < \infty$. A locally integrable weight w is said to be an A_p-weight, if $0 < w < \infty$ almost everywhere, and $A_p(w) \equiv \sup_{Q \in \mathcal{Q}} m_Q(w) m_Q(w^{-\frac{1}{p-1}})^{p-1} < \infty$. The Muckenhoupt class A_p collects all locally integrable weights w for which $A_p(w) < \infty$. It is known that $|\cdot|^\kappa \in A_p$ if and only if $-n < \kappa < n(p-1)$ [2, 8].

We prove the following lemma:

Lemma 3.1. *Let $1 < p < \infty$. Suppose that*

$$-n < \kappa_1 < \kappa_2 < n(p-1), -n < \kappa_3 < 0.$$

Then the weights w and v, given by (1.2) and (1.5), respectively, belong to A_p.

Proof. Since

$$-n < -\frac{\kappa_2}{p-1} < -\frac{\kappa_1}{p-1} < \frac{n}{p-1} = n(p'-1),$$

we see that $|\cdot|^{-\frac{\kappa_1}{p-1}}, |\cdot|^{-\frac{\kappa_2}{p-1}} \in A_{p'}$. Thus, $|\cdot|^{-\frac{\kappa_1}{p-1}} + |\cdot|^{-\frac{\kappa_2}{p-1}} \in A_{p'}$. Hence $w^{-\frac{1}{p-1}} \in A_{p'}$, or equivalently, $w \in A_p$.

If we consider the special case of $\kappa_2 = 0$ in the above, then we see that $v \in A_p$. ■

We prove Theorem 1.2. Choose κ_1, κ_2 so that

$$-n < \kappa_1 < p\alpha < \kappa_2 < n(p-1).$$

It suffices to show that $\mathcal{C}_m$ is bounded on $L^p(w)$ together with the estimate

$$\|\mathcal{C}_m f\|_{L^p(w)} \lesssim \|f\|_{L^p(w)}, \tag{3.1}$$

where the implicit constant in $\lesssim$ depends only on the A_p-constant of w. In fact, once this is done, we have

$$\|\mathcal{C}_m f\|_{\dot{K}^{\alpha,q}_p} \sim \left(\sum_{l=-\infty}^{\infty} (2^{l\alpha}\|\mathcal{C}_m f\|_{L^p(w_l)})^q\right)^{\frac{1}{q}} \lesssim \left(\sum_{l=-\infty}^{\infty} (2^{l\alpha}\|f\|_{L^p(w_l)})^q\right)^{\frac{1}{q}} \sim \|f\|_{\dot{K}^{\alpha,q}_p}$$

for all $f \in L^\infty_{\rm c}(\mathbb{R}^n)$. Here w_l is the weight given by (1.2) and (1.3). We note that $A_p(w_l) = A_p(w_0)$ for all $l \in \mathbb{Z}$. Therefore, matters are reduced to the proof of (3.1).

However, since $\mathcal{C}_m$ is proved to be bounded on $L^p(\mathbb{R}^n)$ for all $1 < p < \infty$ in [7, Theorem 1.1], (3.1) can be proved by using the sharp maximal operator in the same manner as [2, Theorem 6.3.3]. Therefore, the proof of Theorem 1.2 is complete.

4. Proof of Theorem 1.5

We invoke the following fact from [6]:

Lemma 4.1. *Let $0 < \beta < n$, $1 < p_1 < p_2 < \infty$ satisfy $\frac{1}{p_2} = \frac{1}{p_1} - \frac{\beta}{n}$. Let w be a weight such that*

$$w^{-p_1'} \in A_{1+\frac{p_1'}{p_2}}. \tag{4.1}$$

Then I_β maps boundedly from $L^{p_1}(w^{p_1})$ to $L^{p_2}(w^{p_2})$.

With Lemma 4.1 in mind, we prove Theorem 1.5. We can choose κ_1, κ_2 so that

$$-n < \kappa_1 < p_1\alpha < \kappa_2 < n(p_1 - 1)$$

and that

$$-n < -\frac{\kappa_2}{p_1 - 1} < -\frac{\kappa_1}{p_1 - 1} < n\frac{p_1'}{p_2}.$$

Then w, defined by (1.2), satisfies (4.1). Thus, using (1.3) and (2.1), we obtain

$$\begin{aligned}\|I_\beta f\|_{\dot{K}^{\alpha,q}_{p_2}} &\lesssim \left(\sum_{l=-\infty}^{\infty}\left(2^{l\alpha}\|I_\beta f\|_{L^{p_2}\left(w_l^{\frac{p_2}{p_1}}\right)}\right)^q\right)^{\frac{1}{q}}\\ &\lesssim \left(\sum_{l=-\infty}^{\infty}(2^{l\alpha}\|f\|_{L^{p_1}(w_l)})^q\right)^{\frac{1}{q}} \sim \|f\|_{\dot{K}^{\alpha,q}_{p_1}},\end{aligned}$$

proving Theorem 1.5.

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