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General inequalities of the Hilbert integral type using the method of switching to polar coordinates

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Abstract Various inequalities of the Hilbert integral type have been established in the literature using different methods. Among them, the classical Hilbert integral inequality was proved in an elegant way by David C. Ullrich in 2013. It consists of using the method of switching to polar coordinates after some thorough integral manipulations. Despite its effectiveness, this method seems to have been understudied for more of the topic. In this paper, we rehabilitate it somewhat and show how it can be used to prove new general inequalities of the Hilbert integral type, including some with multiple tuning parameters. Particular examples of interest are also discussed.

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1. Introduction

Integral inequalities are fundamental to several branches of mathematics. They have a wide range of applications, providing essential tools for estimating integrals, studying functional spaces, and analyzing the behavior of solutions to differential equations. Among the list of well-known integral inequalities is the Hilbert integral inequality. It is expressed as follows: Given two square-integrable functions $f : [0, +\infty) \to [0, +\infty)$ and $g : [0, +\infty) \to [0, +\infty)$, we have

$$\int_0^{+\infty}\int_0^{+\infty} \frac{f(x)g(y)}{x+y}dxdy \leq \pi\sqrt{\int_0^{+\infty} f^2(x)dx}\sqrt{\int_0^{+\infty} g^2(x)dx}. \tag{1.1}$$

The constant π cannot be improved; it is the best. We refer to [8]. This result has inspired several extensions and generalizations, and still attracts attention. We refer to the studies in [2–5, 11–18, 20]. We also credit the full survey by Qiang Chen and Bicheng Yang published in 2015 in [6] and the references therein. The existing results relevant to the purpose of this paper are presented below.

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- A result established by Bicheng Yang in 1998 in [14, Equation (4)] can be expressed as follows: Given $\beta > 0$, $\epsilon \in (0, 1]$ and two square-integrable functions $f : [0, \beta] \to [0, +\infty)$ and $g : [0, \beta] \to [0, +\infty)$, we have

$$\int_0^\beta \int_0^\beta \frac{f(x)g(y)}{(x+y)^\epsilon} dxdy$$
$$\leq B\left(\frac{\epsilon}{2}, \frac{\epsilon}{2}\right) \sqrt{\int_0^\beta \kappa_{\beta,\epsilon}(x) x^{1-\epsilon} f^2(x) dx} \sqrt{\int_0^\beta \kappa_{\beta,\epsilon}(x) x^{1-\epsilon} g^2(x) dx}, \tag{1.2}$$

where

$$\kappa_{\beta,\epsilon}(x) = 1 - \frac{1}{2}\left(\frac{x}{\beta}\right)^{\epsilon/2}$$

and $B(x, y)$ denotes the standard beta function defined by $B(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt$, with $x > 0$ and $y > 0$. The proof is based on a suitable decomposition of the double integral, the Cauchy-Schwarz integral inequality and several changes of variables of the scale type. This result is interesting because the double integral is taken over a finite domain, i.e., $[0, \beta]^2 = [0, \beta] \times [0, \beta]$. We see how considering this domain affects the classical Hilbert integral inequality recalled in Equation (1.1), and also the addition of the tuning parameter ϵ provides a new level of adaptability, making the result modulable. In fact, the inequality in Equation (1.2) can be extended to $\epsilon > 0$, provided that the integrals on the right-hand side exist. In addition, letting $\beta \to +\infty$, we have $\kappa_{\beta,\epsilon}(x) \to 1$, and we get

$$\int_0^{+\infty} \int_0^{+\infty} \frac{f(x)g(y)}{(x+y)^\epsilon} dxdy$$
$$\leq B\left(\frac{\epsilon}{2}, \frac{\epsilon}{2}\right) \sqrt{\int_0^{+\infty} x^{1-\epsilon} f^2(x) dx} \sqrt{\int_0^{+\infty} x^{1-\epsilon} g^2(x) dx}, \tag{1.3}$$

which is a well-known variant of the Hilbert integral inequality discussed in detail in [19].

- Another important work from our point of view is that of David C. Ullrich in [10]. It is shown how the method of switching to polar coordinates can be used quite easily to prove the classical Hilbert integral inequality. In particular, the polar coordinates introduce an angle variable that varies along $[0, \pi/2]$, and the constant π of this interval fully explains the constant π of the Hilbert integral inequality. Thus, thanks to this approach, we can see how π arises naturally in such an integral setting, with a proof that is concise and easy to understand.

Surprisingly, despite its simplicity and intuitiveness, the method of switching to polar coordinates has, to the best of our knowledge, not been used to establish new inequalities of the Hilbert integral type. This is a gap that this paper aims to fill. More precisely, we use the polar transformation to prove two general inequalities of the Hilbert integral type, which can be seen as variants of the one in Equation (1.2). The first variant considers the integration over a triangle domain, i.e., the one that connects the two variables x and y in the following way: $x + y \leq \beta$, where $\beta > 0$. As far as we know, this is a new setting in this context. The second variant does not innovate in the area of integration but in the complexity of the integrated function. It introduces multiple power functions and a bivariate operator that may not be separable in x and y. We thus considerably generalize the setting of the inequality in Equation (1.2).

The rest of the paper consists of the following sections: Section 2 focuses on our first new Hilbert integral type inequality, with several examples and derived results. Section 3 is the analogue for our second new Hilbert integral type inequality. A conclusion is given in Section 4.

2. Hilbert integral type inequality over a triangle domain

2.1. Main result

Our variant of Equation (1.2), characterized by the integration over a triangle domain, is presented in the theorem below. As mentioned earlier, a key to the proof is the method of switching to polar coordinates, inspired by the methodology in [10].

Theorem 2.1. *Let $\epsilon \geq 0$, $\beta > 0$, and $f : [0, \beta] \to [0, +\infty)$ and $g : [0, \beta] \to [0, +\infty)$ be two functions. Then we have*

$$\int_0^{\beta} \left[\int_0^{\beta - x} \frac{f(x)g(y)}{(x+y)^{\epsilon}} dy \right] dx \leq \sqrt{\int_0^{\beta} \eta_{\beta,\epsilon}(x) x^{1-\epsilon} f^2(x) dx} \sqrt{\int_0^{\beta} \eta_{\beta,\epsilon}(x) x^{1-\epsilon} g^2(x) dx},$$

where

$$\eta_{\beta,\epsilon}(x) = B\left(\frac{\epsilon}{2}, \frac{\epsilon}{2}; 1 - \frac{x}{\beta} \right) \tag{2.1}$$

and $B(x, y; a)$ denotes the incomplete beta function defined by

$$B(x, y; a) = \int_0^a t^{x-1}(1-t)^{y-1} dt,$$

with $x > 0$, $y > 0$ and $a \in [0, 1]$, provided that the integrals on the right-hand side of the inequality exist.

Proof. Let us set

$$I_{\beta,\epsilon} = \int_0^{\beta} \left[\int_0^{\beta - x} \frac{f(x)g(y)}{(x+y)^{\epsilon}} dy \right] dx = \int \int_{\{(x,y)\in[0,+\infty)^2;\ x+y\leq\beta\}} \frac{f(x)g(y)}{(x+y)^{\epsilon}} dxdy.$$

Using the change of variables $(x, y) = (r^2, s^2)$, it can be expressed as follows:

$$I_{\beta,\epsilon} = 4 \int \int_{\{(r,s)\in[0,+\infty)^2;\ r^2+s^2\leq\beta\}} \frac{rsf(r^2)g(s^2)}{(r^2+s^2)^{\epsilon}} drds.$$

We now apply the method of switching to polar coordinates to deal with the denominator term r^2+s^2 (this term was introduced for this purpose only). Consider the polar change of variables $(r, s) = (\rho\cos(\theta), \rho\sin(\theta))$, which is of Jacobian ρ, noting that $r \geq 0$, $s \geq 0$ and $r^2+s^2 \leq \beta$ give $\rho \in [0, \sqrt{\beta}]$ and $\theta \in [0, \pi/2]$, and using the basic formula $\cos^2(\theta)+\sin^2(\theta) =$

1, we get

$$I_{\beta,\epsilon} = 4\int_0^{\pi/2}\int_0^{\sqrt{\beta}} \frac{\rho\cos(\theta)\rho\sin(\theta)f[\rho^2\cos^2(\theta)]g[\rho^2\sin^2(\theta)]}{\left[\rho^2\cos^2(\theta)+\rho^2\sin^2(\theta)\right]^{\epsilon}}\rho d\rho d\theta$$
$$= 4\int_0^{\pi/2}\cos(\theta)\sin(\theta)\int_0^{\sqrt{\beta}} f[\rho^2\cos^2(\theta)]g[\rho^2\sin^2(\theta)]\rho^{3-2\epsilon}d\rho d\theta.$$

Using the Cauchy-Schwarz integral inequality with respect to ρ, we obtain

$$I_{\beta,\epsilon} \leq 4\int_0^{\pi/2}\cos(\theta)\sin(\theta)\times$$
$$\sqrt{\int_0^{\sqrt{\beta}} f^2[\rho^2\cos^2(\theta)]\rho^{3-2\epsilon}d\rho}\sqrt{\int_0^{\sqrt{\beta}} g^2[\rho^2\sin^2(\theta)]\rho^{3-2\epsilon}d\rho}d\theta$$
$$= 4\int_0^{\pi/2}\cos(\theta)\sin(\theta)\times$$
$$\sqrt{\int_0^{\sqrt{\beta}} f^2[\rho_1^2\cos^2(\theta)]\rho_1^{3-2\epsilon}d\rho_1}\sqrt{\int_0^{\sqrt{\beta}} g^2[\rho_2^2\sin^2(\theta)]\rho_2^{3-2\epsilon}d\rho_2}d\theta. \quad (2.2)$$

Applying the changes of variables $u = \rho_1^2\cos^2(\theta)$ with respect to ρ_1, so that $\rho_1 d\rho_1 = 1/[2\cos^2(\theta)]du$, and $v = \rho_2^2\sin^2(\theta)$ with respect to ρ_2, so that $\rho_2 d\rho_2 = 1/[2\sin^2(\theta)]dv$, we get

$$4\int_0^{\pi/2}\cos(\theta)\sin(\theta)\sqrt{\int_0^{\beta\cos^2(\theta)} f^2(u)\left[\frac{u}{\cos^2(\theta)}\right]^{1-\epsilon}\frac{1}{2\cos^2(\theta)}du}\times$$
$$\sqrt{\int_0^{\beta\sin^2(\theta)} g^2(v)\left[\frac{v}{\sin^2(\theta)}\right]^{1-\epsilon}\frac{1}{2\sin^2(\theta)}dv}d\theta$$
$$= 2\int_0^{\pi/2}\cos^{\epsilon-1}(\theta)\sin^{\epsilon-1}(\theta)\times$$
$$\sqrt{\int_0^{\beta\cos^2(\theta)} u^{1-\epsilon}f^2(u)du}\sqrt{\int_0^{\beta\sin^2(\theta)} v^{1-\epsilon}g^2(v)dv}d\theta. \quad (2.3)$$

Using the Cauchy-Schwarz integral inequality with respect to θ, we have

$$2\int_0^{\pi/2}\cos^{\epsilon-1}(\theta)\sin^{\epsilon-1}(\theta)\sqrt{\int_0^{\beta\cos^2(\theta)} u^{1-\epsilon}f^2(u)du}\sqrt{\int_0^{\beta\sin^2(\theta)} v^{1-\epsilon}g^2(v)dv}d\theta$$
$$\leq 2\sqrt{J_{\beta,\epsilon}}\sqrt{K_{\beta,\epsilon}}. \quad (2.4)$$

where

$$J_{\beta,\epsilon} = \int_0^{\pi/2}\cos^{\epsilon-1}(\theta)\sin^{\epsilon-1}(\theta)\left[\int_0^{\beta\cos^2(\theta)} u^{1-\epsilon}f^2(u)du\right]d\theta$$

and

$$K_{\beta,\epsilon} = \int_0^{\pi/2}\cos^{\epsilon-1}(\theta)\sin^{\epsilon-1}(\theta)\left[\int_0^{\beta\sin^2(\theta)} v^{1-\epsilon}g^2(v)dv\right]d\theta.$$

Let us now determine $J_{\beta,\epsilon}$ and $K_{\beta,\epsilon}$ one after the other. For both, we plan to use a change in the order of integration, as described in [9, p. 307].

Starting with $J_{\beta,\epsilon}$, all the main functions involved are positive, so by changing the order of integration, taking into account that $\cos^2(\theta)$ is decreasing for $\theta \in [0, \pi/2]$, we get

$$\begin{aligned} J_{\beta,\epsilon} &= \int_0^\beta \left[\int_0^{\arccos\left[\sqrt{u/\beta}\right]} \cos^{\epsilon-1}(\theta)\sin^{\epsilon-1}(\theta) u^{1-\epsilon} f^2(u) d\theta \right] du \\ &= \int_0^\beta \left[u^{1-\epsilon} f^2(u) \int_0^{\arccos\left[\sqrt{u/\beta}\right]} \cos^{\epsilon-1}(\theta)\sin^{\epsilon-1}(\theta) d\theta \right] du. \end{aligned}$$

Considering the change of variables $t = \sin^2(\theta)$, so that $dt = 2\sin(\theta)\cos(\theta)d\theta$, $\sin^2\left\{\arccos\left[\sqrt{u/\beta}\right]\right\} = 1 - u/\beta$ and $\cos^2(\theta) = 1 - \sin^2(\theta)$, we obtain

$$\begin{aligned} &\int_0^{\arccos\left[\sqrt{u/\beta}\right]} \cos^{\epsilon-1}(\theta)\sin^{\epsilon-1}(\theta) d\theta \\ &= \frac{1}{2} \int_0^{\sin^2\left\{\arccos\left[\sqrt{u/\beta}\right]\right\}} [1 - \sin^2(\theta)]^{\epsilon/2-1} [\sin^2(\theta)]^{\epsilon/2-1} [2\sin(\theta)\cos(\theta)] d\theta \\ &= \frac{1}{2} \int_0^{1-u/\beta} (1-t)^{\epsilon/2-1} t^{\epsilon/2-1} dt = \frac{1}{2} B\left(\frac{\epsilon}{2}, \frac{\epsilon}{2}; 1 - \frac{u}{\beta}\right). \end{aligned}$$

We thus find that

$$J_{\beta,\epsilon} = \frac{1}{2} \int_0^\beta B\left(\frac{\epsilon}{2}, \frac{\epsilon}{2}; 1 - \frac{u}{\beta}\right) u^{1-\epsilon} f^2(u) du. \tag{2.5}$$

On the other hand, for $K_{\beta,\epsilon}$, by changing the order of integration, taking into account that $\sin^2(\theta)$ is increasing for $\theta \in [0, \pi/2]$, we get

$$\begin{aligned} K_{\beta,\epsilon} &= \int_0^\beta \left[\int_{\arcsin\left[\sqrt{v/\beta}\right]}^{\pi/2} \cos^{\epsilon-1}(\theta)\sin^{\epsilon-1}(\theta) v^{1-\epsilon} g^2(v) d\theta \right] dv \\ &= \int_0^\beta \left[v^{1-\epsilon} g^2(v) \int_{\arcsin\left[\sqrt{v/\beta}\right]}^{\pi/2} \cos^{\epsilon-1}(\theta)\sin^{\epsilon-1}(\theta) d\theta \right] dv. \end{aligned}$$

Using the change of variables $t = \sin^2(\theta)$, so that $dt = 2\sin(\theta)\cos(\theta)d\theta$,

$\sin^2\left\{\arcsin\left[\sqrt{v/\beta}\right]\right\} = v/\beta$ and $\cos^2(\theta) = 1-\sin^2(\theta)$, and the change of variables $w = 1-t$, we have

$$\begin{aligned}
&\int_{\arcsin\left[\sqrt{v/\beta}\right]}^{\pi/2} \cos^{\epsilon-1}(\theta)\sin^{\epsilon-1}(\theta)d\theta \\
&= \frac{1}{2}\int_{\sin^2\left\{\arcsin\left[\sqrt{v/\beta}\right]\right\}}^{1} [1-\sin^2(\theta)]^{\epsilon/2-1}[\sin^2(\theta)]^{\epsilon/2-1}[2\sin(\theta)\cos(\theta)]d\theta \\
&= \frac{1}{2}\int_{v/\beta}^{1}(1-t)^{\epsilon/2-1}t^{\epsilon/2-1}dt = \frac{1}{2}\int_0^{1-v/\beta} w^{\epsilon/2-1}(1-w)^{\epsilon/2-1}dw \\
&= \frac{1}{2}B\left(\frac{\epsilon}{2},\frac{\epsilon}{2};1-\frac{v}{\beta}\right).
\end{aligned}$$

We thus obtain

$$K_{\beta,\epsilon} = \frac{1}{2}\int_0^{\beta} B\left(\frac{\epsilon}{2},\frac{\epsilon}{2};1-\frac{v}{\beta}\right)v^{1-\epsilon}g^2(v)dv. \tag{2.6}$$

Combining Equations (2.2), (2.3), (2.4), (2.5) and (2.6), simplifying the constants 2, and standardizing the names of the variables, i.e., "$x = u$" and "$x = v$", we establish that

$$I_{\beta,\epsilon} \le \sqrt{\int_0^{\beta}\eta_{\beta,\epsilon}(x)x^{1-\epsilon}f^2(x)dx}\sqrt{\int_0^{\beta}\eta_{\beta,\epsilon}(x)x^{1-\epsilon}g^2(x)dx},$$

where

$$\eta_{\beta,\epsilon}(x) = B\left(\frac{\epsilon}{2},\frac{\epsilon}{2};1-\frac{x}{\beta}\right).$$

This concludes the proof of Theorem 2.1. ■

Some special cases of Theorem 2.1 are now discussed. Taking $\epsilon = 1$, we get

$$\int_0^{\beta}\left[\int_0^{\beta-x}\frac{f(x)g(y)}{x+y}dy\right]dx \le \sqrt{\int_0^{\beta}\eta_{\beta,1}(x)f^2(x)dx}\sqrt{\int_0^{\beta}\eta_{\beta,1}(x)g^2(x)dx},$$

where

$$\eta_{\beta,1}(x) = B\left(\frac{1}{2},\frac{1}{2};1-\frac{x}{\beta}\right) = 2\arcsin\left[\sqrt{1-\frac{x}{\beta}}\right] = 2\arccos\left[\sqrt{\frac{x}{\beta}}\right].$$

This gives the following new, simpler integral inequality:

$$\begin{aligned}
&\int_0^{\beta}\left[\int_0^{\beta-x}\frac{f(x)g(y)}{x+y}dy\right]dx \\
&\le 2\sqrt{\int_0^{\beta}\arccos\left[\sqrt{\frac{x}{\beta}}\right]f^2(x)dx}\sqrt{\int_0^{\beta}\arccos\left[\sqrt{\frac{x}{\beta}}\right]g^2(x)dx}.
\end{aligned}$$

It is of mathematical interest to see how the arccosine function appears in such an integral inequality setting, which is quite rare in the field of Hilbert integral inequality types. Since

$\arccos(0) = \pi/2$, by applying $\beta \to +\infty$, we also find that

$$\int_0^{+\infty} \left[\int_0^{+\infty} \frac{f(x)g(y)}{x+y} dy\right] dx \leq 2\sqrt{\int_0^{+\infty} \frac{\pi}{2} f^2(x)dx}\sqrt{\int_0^{+\infty} \frac{\pi}{2} g^2(x)dx}$$
$$= \pi\sqrt{\int_0^{+\infty} f^2(x)dx}\sqrt{\int_0^{+\infty} g^2(x)dx},$$

which corresponds to the classical Hilbert integral inequality as recalled in Equation (1.1).

Selecting $\epsilon = 2$, we get

$$\int_0^{\beta} \left[\int_0^{\beta - x} \frac{f(x)g(y)}{(x+y)^2} dy\right] dx$$
$$\leq \sqrt{\int_0^{\beta} \eta_{\beta,2}(x)x^{-1} f^2(x)dx}\sqrt{\int_0^{\beta} \eta_{\beta,2}(x)x^{-1} g^2(x)dx},$$

where

$$\eta_{\beta,2}(x) = B\left(1, 1; 1 - \frac{x}{\beta}\right) = 1 - \frac{x}{\beta}.$$

With reference to the upper bound established in [14, Equation (4)], also recalled in Equation (1.2), note that, for any $x \in [0, \beta]$, we have

$$\eta_{\beta,2}(x) \leq \kappa_{\beta,2}(x).$$

This implies that the upper bound obtained is lower than that in Equation (1.2), as expected given the sharpness of the techniques used and the fact that

$$\int_0^{\beta} \left[\int_0^{\beta - x} \frac{f(x)g(y)}{(x+y)^2} dy\right] dx \leq \int_0^{\beta}\int_0^{\beta} \frac{f(x)g(y)}{(x+y)^2} dydx.$$

Taking $\epsilon = 3$, we obtain

$$\int_0^{\beta} \left[\int_0^{\beta - x} \frac{f(x)g(y)}{(x+y)^3} dy\right] dx$$
$$\leq \sqrt{\int_0^{\beta} \eta_{\beta,3}(x)x^{-2} f^2(x)dx}\sqrt{\int_0^{\beta} \eta_{\beta,3}(x)x^{-2} g^2(x)dx},$$

where

$$\eta_{\beta,3}(x) = B\left(\frac{3}{2}, \frac{3}{2}; 1 - \frac{x}{\beta}\right) = \frac{1}{4}\left\{\left(1 - 2\frac{x}{\beta}\right)\sqrt{\frac{x}{\beta}\left(1 - \frac{x}{\beta}\right)} + \arccos\left[\sqrt{\frac{x}{\beta}}\right]\right\}.$$

As a last example, selecting $\epsilon = 4$, we get

$$\int_0^{\beta} \left[\int_0^{\beta - x} \frac{f(x)g(y)}{(x+y)^4} dy\right] dx$$
$$\leq \sqrt{\int_0^{\beta} \eta_{\beta,4}(x)x^{-3} f^2(x)dx}\sqrt{\int_0^{\beta} \eta_{\beta,4}(x)x^{-3} g^2(x)dx},$$

where

$$\eta_{\beta,4}(x) = B\left(2, 2; 1 - \frac{x}{\beta}\right) = \frac{1}{6}\left(1 + 2\frac{x}{\beta}\right)\left(1 - \frac{x}{\beta}\right)^2.$$

All the above inequalities are, to the best of our knowledge, new inequalities of the Hilbert integral type in the literature.

2.2. Secondary results

Some secondary results related to Theorem 2.1 are now presented. The proposition below presents an integral type inequality involving only one adaptable function and still an integration over a triangle domain.

Proposition 2.2. *Let $\epsilon \in (0, 1)$, $\beta > 0$, and $f : [0, \beta] \to [0, +\infty)$ be a function. Then we have*

$$\int_0^\beta \left[\int_0^{\beta - x} \frac{f(y)}{(x + y)^\epsilon} dy\right]^2 dx \leq A_{\beta,\epsilon} \int_0^\beta \eta_{\beta,\epsilon}(x) x^{1-\epsilon} f^2(x) dx,$$

where $\eta_{\beta,\epsilon}(x)$ is given by Equation (2.1) and

$$A_{\beta,\epsilon} = \sup_{x \in [0,\beta]} \left[\eta_{\beta,\epsilon}(x) x^{1-\epsilon}\right],$$

which exists because $\eta_{\beta,\epsilon}(x) x^{1-\epsilon}$ is a continuous function with respect to x, provided that the integrals on the right-hand side of the inequality exist.

Proof. Let us consider the following function based on $f(y)$:

$$h(x) = \int_0^{\beta - x} \frac{f(y)}{(x + y)^\epsilon} dy.$$

Applying the Fubini-Tonelli integral theorem, the general statement of which is recalled in the appendix, we get

$$\int_0^\beta \left[\int_0^{\beta - x} \frac{f(y)}{(x + y)^\epsilon} dy\right]^2 dx = \int_0^\beta \left[\int_0^{\beta - x} \frac{f(y)}{(x + y)^\epsilon} dy\right] h(x) dx$$
$$= \int_0^\beta \left[\int_0^{\beta - x} \frac{h(x) f(y)}{(x + y)^\epsilon} dy\right] dx.$$

Theorem 2.1 applied with the appropriate functions, i.e., "$f(x) = h(x)$" and "$g(y) = f(y)$", and the fact that, for any $\epsilon \in (0, 1)$ and $x \in [0, \beta]$, we have $\eta_{\beta,\epsilon}(x) x^{1-\epsilon} \leq A_{\beta,\epsilon}$, give

$$\int_0^\beta \left[\int_0^{\beta-x} \frac{h(x)f(y)}{(x+y)^\epsilon}dy\right]dx$$
$$\leq \sqrt{\int_0^\beta \eta_{\beta,\epsilon}(x)x^{1-\epsilon}h^2(x)dx}\sqrt{\int_0^\beta \eta_{\beta,\epsilon}(x)x^{1-\epsilon}f^2(x)dx}$$
$$= \sqrt{\int_0^\beta \eta_{\beta,\epsilon}(x)x^{1-\epsilon}\left[\int_0^{\beta-x} \frac{f(y)}{(x+y)^\epsilon}dy\right]^2dx}\sqrt{\int_0^\beta \eta_{\beta,\epsilon}(x)x^{1-\epsilon}f^2(x)dx}$$
$$\leq \sqrt{A_{\beta,\epsilon}}\sqrt{\int_0^\beta \left[\int_0^{\beta-x} \frac{f(y)}{(x+y)^\epsilon}dy\right]^2dx}\sqrt{\int_0^\beta \eta_{\beta,\epsilon}(x)x^{1-\epsilon}f^2(x)dx}.$$

Therefore, we have

$$\int_0^\beta \left[\int_0^{\beta-x} \frac{f(y)}{(x+y)^\epsilon}dy\right]^2dx$$
$$\leq \sqrt{A_{\beta,\epsilon}}\sqrt{\int_0^\beta \left[\int_0^{\beta-x} \frac{f(y)}{(x+y)^\epsilon}dy\right]^2dx}\sqrt{\int_0^\beta \eta_{\beta,\epsilon}(x)x^{1-\epsilon}f^2(x)dx}.$$

If we arrange the common terms on both sides and consider the square exponent, we obtain

$$\int_0^\beta \left[\int_0^{\beta-x} \frac{f(y)}{(x+y)^\epsilon}dy\right]^2dx \leq A_{\beta,\epsilon}\int_0^\beta \eta_{\beta,\epsilon}(x)x^{1-\epsilon}f^2(x)dx.$$

The desired result is obtained, completing the proof of Proposition 2.2. ■

This result is the basis for the further analysis of the specific subjacent integral operator defined over a triangle domain. It must be considered as a first step in this direction.

In the proposition below, we propose an alternative integral inequality to the one in Theorem 2.1, but with some different assumptions on ϵ and the underlying integral conditions.

Proposition 2.3. *Let $\epsilon \in \mathbb{R}$, $\beta > 0$, and $f : [0,\beta] \to [0,+\infty)$ and $g : [0,\beta] \to [0,+\infty)$ be two functions. Then we have*

$$\int_0^\beta \left[\int_0^{\beta-x} \frac{f(x)g(y)}{(x+y)^\epsilon}dy\right]dx$$
$$\leq \sqrt{\int_0^\beta \psi_{\beta,\epsilon}(x)x^{1-\epsilon}f^2(x)dx}\sqrt{\int_0^\beta \psi_{\beta,\epsilon}(x)x^{1-\epsilon}g^2(x)dx},$$

where

$$\psi_{\beta,\epsilon}(x) = \begin{cases} \frac{1}{1-\epsilon}\left[\left(\frac{\beta}{x}\right)^{1-\epsilon} - 1\right] & \text{if } \epsilon \in \mathbb{R}/\{1\} \\ \log\left(\frac{\beta}{x}\right) & \text{if } \epsilon = 1 \end{cases}, \tag{2.7}$$

provided that the integrals on the right-hand side of the inequality exist.

Proof. Let us set $\mathcal{A}_\beta = \{(x,y) \in [0,+\infty)^2;\ x+y \leq \beta\}$. It follows from the Cauchy-Schwarz integral inequality applied with respect to x and y, and the Fubini-Tonelli integral theorem that

$$\begin{aligned}
&\int_0^\beta \left[\int_0^{\beta-x} \frac{f(x)g(y)}{(x+y)^\epsilon}dy\right]dx = \int\int_{\mathcal{A}_\beta} \frac{f(x)g(y)}{(x+y)^\epsilon}dydx \\
&= \int\int_{\mathcal{A}_\beta} \frac{f(x)}{(x+y)^{\epsilon/2}} \times \frac{g(y)}{(x+y)^{\epsilon/2}}dydx \\
&\leq \sqrt{\int\int_{\mathcal{A}_\beta} \frac{f^2(x)}{(x+y)^\epsilon}dydx}\sqrt{\int\int_{\mathcal{A}_\beta} \frac{g^2(y)}{(x+y)^\epsilon}dydx} \\
&= \sqrt{\int_0^\beta \left[\int_0^{\beta-x} \frac{1}{(x+y)^\epsilon}dy\right] f^2(x)dx}\sqrt{\int_0^\beta \left[\int_0^{\beta-y} \frac{1}{(x+y)^\epsilon}dx\right] g^2(y)dy.}
\end{aligned}$$

For any $\epsilon \neq 1$ and $x \in (0,\beta)$, we have

$$\begin{aligned}
\int_0^{\beta-x} \frac{1}{(x+y)^\epsilon}dy &= \left[\frac{1}{1-\epsilon}(x+y)^{1-\epsilon}\right]_{y=0}^{y=\beta-x} = \frac{1}{1-\epsilon}\left(\beta^{1-\epsilon} - x^{1-\epsilon}\right) \\
&= \psi_{\beta,\epsilon}(x)x^{1-\epsilon},
\end{aligned}$$

where $\psi_{\beta,\epsilon}(x)$ is given in Equation (2.7). For the case $\epsilon = 1$, by just editing the integrated function, we have

$$\int_0^{\beta-x} \frac{1}{x+y}dy = [\log(x+y)]_{y=0}^{y=\beta-x} = \log\left(\frac{\beta}{x}\right) = \psi_{\beta,\epsilon}(x)x^{1-\epsilon}.$$

Proceeding in a similar manner, for any $\epsilon \in \mathbb{R}$ and $y \in (0,\beta)$, we have

$$\int_0^{\beta-y} \frac{1}{(x+y)^\epsilon}dx = \psi_{\beta,\epsilon}(y)y^{1-\epsilon}.$$

The combination of the above inequalities and expressions gives

$$\begin{aligned}
&\int_0^\beta \left[\int_0^{\beta-x} \frac{f(x)g(y)}{(x+y)^\epsilon}dy\right]dx \\
&\leq \sqrt{\int_0^\beta \psi_{\beta,\epsilon}(x)x^{1-\epsilon}f^2(x)dx}\sqrt{\int_0^\beta \psi_{\beta,\epsilon}(y)y^{1-\epsilon}g^2(y)dy.}
\end{aligned}$$

Standardizing the notations x and y yields the desired result. This concludes the proof. ∎

In full generality, for given functions f and g, the inequalities in Theorem 2.1 and Proposition 2.3 are difficult to compare because they deal with different assumptions on ϵ and different underlying integral conditions on the functions involved. However, the Hilbert integral inequality is not recovered in Proposition 2.3; this result cannot be considered as a proper generalisation.

3. A "very general" Hilbert integral type inequality

3.1. Main result

Our variant of Equation (1.2), characterized by a high level of generality, is presented in the theorem below. As mentioned earlier, a key to the proof is the method of switching to polar coordinates.

Theorem 3.1. *Let $\epsilon \geq 0$, $\tau \geq 0$, $\sigma \geq 0$, $f : [0,+\infty) \to [0,+\infty)$ and $g : [0,+\infty) \to [0,+\infty)$ be two functions, and $T : [0,+\infty)^2 \to [0,+\infty)$ be a bivariate function satisfying the homogeneous property of degree 0, formulated as follows: For any $\lambda > 0$, we have*

$$T(\lambda x, \lambda y) = T(x, y).$$

Then we have

$$\int_0^{+\infty}\int_0^{+\infty} \frac{f(x)g(y)}{(x+y)^\epsilon} T(x,y)x^\tau y^\sigma dxdy$$
$$\leq C_{\epsilon,\tau,\sigma,[T]}\sqrt{\int_0^{+\infty} x^{\tau+\sigma-\epsilon+1} f^2(x)dx}\sqrt{\int_0^{+\infty} x^{\tau+\sigma-\epsilon+1} g^2(x)dx},$$

where

$$C_{\epsilon,\tau,\sigma,[T]} = \int_0^1 (1-t)^{(\tau-\sigma+\epsilon)/2-1} t^{(\sigma-\tau+\epsilon)/2-1} T(1-t,t)dt,$$

provided that the integrals on the right-hand side of the inequality exist.

Proof. For this proof, we set

$$L_{\epsilon,\tau,\sigma,[T]} = \int_0^{+\infty}\int_0^{+\infty} \frac{f(x)g(y)}{(x+y)^\epsilon} T(x,y)x^\tau y^\sigma dxdy.$$

By using the change of variables $(x, y) = (r^2, s^2)$, we obtain

$$L_{\epsilon,\tau,\sigma,[T]} = 4\int_0^{+\infty}\int_0^{+\infty} \frac{f(r^2)g(s^2)}{(r^2+s^2)^\epsilon} T(r^2,s^2) r^{2\tau+1} s^{2\sigma+1} drds.$$

We now apply the method of switching to polar coordinates to deal with the denominator term $r^2 + s^2$ (this term was introduced for this purpose only). Consider the polar change of variables $(r, s) = (\rho\cos(\theta), \rho\sin(\theta))$, which is of Jacobian ρ, and the homogeneous property of degree 0 of $T(x, y)$ with $\lambda = \rho^2$, we get

$$L_{\epsilon,\tau,\sigma,[T]} = 4\int_0^{\pi/2}\int_0^{+\infty} \frac{f[\rho^2\cos^2(\theta)]g[\rho^2\sin^2(\theta)]}{\left[\rho^2\cos^2(\theta)+\rho^2\sin^2(\theta)\right]^\epsilon} T\left[\rho^2\cos^2(\theta), \rho^2\sin^2(\theta)\right] \times$$
$$[\rho\cos(\theta)]^{2\tau+1}[\rho\sin(\theta)]^{2\sigma+1}\rho d\rho d\theta$$
$$= 4\int_0^{\pi/2} \cos^{2\tau+1}(\theta)\sin^{2\sigma+1}(\theta) T\left[\cos^2(\theta), \sin^2(\theta)\right] M_{\epsilon,\tau,\sigma}(\theta)d\theta, \tag{3.1}$$

where

$$M_{\epsilon,\tau,\sigma}(\theta) = \int_0^{+\infty} f[\rho^2\cos^2(\theta)]g[\rho^2\sin^2(\theta)]\rho^{2(\tau+\sigma-\epsilon)+3}d\rho.$$

Let us now bound this last integral term. Using the Cauchy-Schwarz integral inequality with respect to ρ, we obtain

$$\begin{aligned}
&M_{\epsilon,\tau,\sigma}(\theta)\\
&\leq \sqrt{\int_0^{+\infty} f^2[\rho^2\cos^2(\theta)]\rho^{2(\tau+\sigma-\epsilon)+3}d\rho}\sqrt{\int_0^{+\infty} g^2[\rho^2\sin^2(\theta)]\rho^{2(\tau+\sigma-\epsilon)+3}d\rho}\\
&= \sqrt{\int_0^{+\infty} f^2[\rho_1^2\cos^2(\theta)]\rho_1^{2(\tau+\sigma-\epsilon)+3}d\rho_1}\sqrt{\int_0^{+\infty} g^2[\rho_2^2\sin^2(\theta)]\rho_2^{2(\tau+\sigma-\epsilon)+3}d\rho_2}.
\end{aligned}$$

Applying the changes of variables $u = \rho_1^2\cos^2(\theta)$ with respect to ρ_1, so that $\rho_1 d\rho_1 = 1/[2\cos^2(\theta)]du$, and $v = \rho_2^2\sin^2(\theta)$ with respect to ρ_2, so that $\rho_2 d\rho_2 = 1/[2\sin^2(\theta)]dv$, we get

$$\begin{aligned}
&\sqrt{\int_0^{+\infty} f^2[\rho_1^2\cos^2(\theta)]\rho_1^{2(\tau+\sigma-\epsilon)+3}d\rho_1}\sqrt{\int_0^{+\infty} g^2[\rho_2^2\sin^2(\theta)]\rho_2^{2(\tau+\sigma-\epsilon)+3}d\rho_2}\\
&= \sqrt{\int_0^{+\infty} f^2(u)\left[\frac{u}{\cos^2(\theta)}\right]^{\tau+\sigma-\epsilon+1}\frac{1}{2\cos^2(\theta)}du}\times\\
&\quad\sqrt{\int_0^{+\infty} g^2(v)\left[\frac{v}{\sin^2(\theta)}\right]^{\tau+\sigma-\epsilon+1}\frac{1}{2\sin^2(\theta)}dv}\\
&= \frac{1}{2\cos^{\tau+\sigma-\epsilon+2}(\theta)\sin^{\tau+\sigma-\epsilon+2}(\theta)}\times\\
&\quad\sqrt{\int_0^{+\infty} u^{\tau+\sigma-\epsilon+1}f^2(u)du}\sqrt{\int_0^{+\infty} v^{\tau+\sigma-\epsilon+1}g^2(v)dv}.
\end{aligned}$$

Hence, we have

$$\begin{aligned}
M_{\epsilon,\tau,\sigma}(\theta) \leq{}& \frac{1}{2\cos^{\tau+\sigma-\epsilon+2}(\theta)\sin^{\tau+\sigma-\epsilon+2}(\theta)}\times\\
&\sqrt{\int_0^{+\infty} u^{\tau+\sigma-\epsilon+1}f^2(u)du}\sqrt{\int_0^{+\infty} v^{\tau+\sigma-\epsilon+1}g^2(v)dv}. \qquad (3.2)
\end{aligned}$$

Therefore, combining Equations (3.1) and (3.2), isolating the integral with respect to θ and adopting the notations as "$u = x$" and "$v = x$" for the sake of uniformization, we get

$$L_{\epsilon,\tau,\sigma,[T]} \leq 4\int_0^{\pi/2} \cos^{2\tau+1}(\theta)\sin^{2\sigma+1}(\theta)T\left[\cos^2(\theta),\sin^2(\theta)\right] \times \frac{1}{2\cos^{\tau+\sigma-\epsilon+2}(\theta)\sin^{\tau+\sigma-\epsilon+2}(\theta)} \times \sqrt{\int_0^{+\infty} u^{\tau+\sigma-\epsilon+1}f^2(u)du}\sqrt{\int_0^{+\infty} v^{\tau+\sigma-\epsilon+1}g^2(v)dv}d\theta$$
$$= N_{\epsilon,\tau,\sigma,[T]}\sqrt{\int_0^{+\infty} x^{\tau+\sigma-\epsilon+1}f^2(x)dx}\sqrt{\int_0^{+\infty} x^{\tau+\sigma-\epsilon+1}g^2(x)dx},$$

where

$$N_{\epsilon,\tau,\sigma,[T]} = 2\int_0^{\pi/2} \cos^{\tau-\sigma+\epsilon-1}(\theta)\sin^{\sigma-\tau+\epsilon-1}(\theta)T\left[\cos^2(\theta),\sin^2(\theta)\right]d\theta.$$

Applying the change of variables $t = \sin^2(\theta)$, so that $dt = 2\sin(\theta)\cos(\theta)d\theta$ and $\cos^2(\theta) = 1 - \sin^2(\theta)$, we obtain

$$N_{\epsilon,\tau,\sigma,[T]} = \int_0^{\pi/2} [1-\sin^2(\theta)]^{(\tau-\sigma+\epsilon)/2-1}[\sin^2(\theta)]^{(\sigma-\tau+\epsilon)/2-1}T\left[1-\sin^2(\theta),\sin^2(\theta)\right] \times [2\sin(\theta)\cos(\theta)]d\theta$$
$$= \int_0^1 (1-t)^{(\tau-\sigma+\epsilon)/2-1}t^{(\sigma-\tau+\epsilon)/2-1}T(1-t,t)dt = C_{\epsilon,\tau,\sigma,[T]}.$$

This concludes the proof of Theorem 3.1. ∎

The generality of Theorem 3.1 is thus characterized by the presence of the tuning parameters ϵ, τ, σ, and the operator $T(x,y)$, which can perturb the separability in x and y of the main integrated function. Note that the homogeneous property of degree 0 assumed for this operator includes the following functional form:

$$T(x,y) = k\left(\frac{x}{y}\right),$$

where $k : [0,+\infty) \to [0,+\infty)$. See, for example, [1].

Some precise examples of Theorem 3.1 are described below.

Example 1. Taking $T(x,y) = 1$, we have

$$\int_0^{+\infty}\int_0^{+\infty} \frac{f(x)g(y)}{(x+y)^\epsilon}x^\tau y^\sigma dxdy$$
$$\leq C_{\epsilon,\tau,\sigma,[T]}\sqrt{\int_0^{+\infty} x^{\tau+\sigma-\epsilon+1}f^2(x)dx}\sqrt{\int_0^{+\infty} x^{\tau+\sigma-\epsilon+1}g^2(x)dx},$$

where

$$C_{\epsilon,\tau,\sigma,[T]} = \int_0^1 (1-t)^{(\tau-\sigma+\epsilon)/2-1} t^{(\sigma-\tau+\epsilon)/2-1} dt$$
$$= B\left[\frac{1}{2}(\tau-\sigma+\epsilon), \frac{1}{2}(\sigma-\tau+\epsilon)\right].$$

Note that the following assumptions on the parameters are required: $\sigma - \tau + \epsilon > 0$ and $\tau - \sigma + \epsilon > 0$. In particular, taking $\tau = 0$ and $\sigma = 0$, which implies that $\epsilon > 0$, we obtain

$$\int_0^{+\infty}\int_0^{+\infty} \frac{f(x)g(y)}{(x+y)^\epsilon} dxdy \leq C_{\epsilon,0,0,[T]} \sqrt{\int_0^{+\infty} x^{1-\epsilon} f^2(x)dx} \sqrt{\int_0^{+\infty} x^{1-\epsilon} g^2(x)dx},$$

where

$$C_{\epsilon,0,0,[T]} = \int_0^1 (1-t)^{\epsilon/2-1} t^{\epsilon/2-1} dt = B\left(\frac{\epsilon}{2}, \frac{\epsilon}{2}\right).$$

We thus refind a result established in [19], recalled in Equation (1.3).

Example 2. As another simple example, let us choose

$$T(x,y) = \frac{\sqrt{xy}}{x+y}.$$

Then it is clear that $T(x,y)$ is homogeneous of degree 0, and Theorem 3.1 implies that

$$\int_0^{+\infty}\int_0^{+\infty} \frac{f(x)g(y)}{(x+y)^\epsilon} \times \frac{\sqrt{xy}}{x+y} x^\tau y^\sigma dxdy$$
$$\leq C_{\epsilon,\tau,\sigma,[T]} \sqrt{\int_0^{+\infty} x^{\tau+\sigma-\epsilon+1} f^2(x)dx} \sqrt{\int_0^{+\infty} x^{\tau+\sigma-\epsilon+1} g^2(x)dx},$$

where

$$C_{\epsilon,\tau,\sigma,[T]} = \int_0^1 (1-t)^{(\tau-\sigma+\epsilon)/2-1} t^{(\sigma-\tau+\epsilon)/2-1} \sqrt{t(1-t)} dt$$
$$= \int_0^1 (1-t)^{(\tau-\sigma+\epsilon+1)/2-1} t^{(\sigma-\tau+\epsilon+1)/2-1} dt$$
$$= B\left[\frac{1}{2}(\tau-\sigma+\epsilon+1), \frac{1}{2}(\sigma-\tau+\epsilon+1)\right].$$

This result is consistent with the application of Theorem 3.1 in Example 1, under the following remark:

$$\int_0^{+\infty}\int_0^{+\infty} \frac{f(x)g(y)}{(x+y)^\epsilon} \times \frac{\sqrt{xy}}{x+y} x^\tau y^\sigma dxdy = \int_0^{+\infty}\int_0^{+\infty} \frac{f(x)g(y)}{(x+y)^{\epsilon_\circ}} x^{\tau_\circ} y^{\sigma_\circ} dxdy,$$

with $\epsilon_\circ = \epsilon + 1$, $\tau_\circ = \tau + 1/2$ and $\sigma_\circ = \sigma + 1/2$.

Example 3. A more sophisticated example is now developed. For any $a > 0$, $b > 0$, $c > 0$ and $d > 0$, let us consider

$$T(x,y) = \frac{x^c y^d}{(ax+by)^{c+d}}.$$

Then it is clear that $T(x, y)$ is positive and homogeneous of degree 0. In the setting of Theorem 3.1, let us set $\tau = \sigma = 0$ and $\epsilon = 0$. Based on this theorem, we find that

$$\int_0^{+\infty}\int_0^{+\infty} f(x)g(y)\frac{x^c y^d}{(ax+by)^{c+d}}dxdy \leq C_{0,0,0,[T]}\sqrt{\int_0^{+\infty} xf^2(x)dx}\sqrt{\int_0^{+\infty} xg^2(x)dx},$$

where

$$C_{0,0,0,[T]} = \int_0^1 (1-t)^{-1}t^{-1}\frac{(1-t)^c t^d}{[a(1-t)+bt]^{c+d}}dt = \int_0^1 \frac{(1-t)^{c-1}t^{d-1}}{[a(1-t)+bt]^{c+d}}dt.$$

In order to deal with a precise referenced integral formula, we do the change of variables $t = \sin^2(\theta)$. We thus obtain

$$C_{0,0,0,[T]} = 2\int_0^{\pi/2} \frac{\cos^{2c-1}(\theta)\sin^{2d-1}(\theta)}{[a\cos^2(\theta)+b\sin^2(\theta)]^{c+d}}d\theta.$$

Since $\tau - \sigma + \epsilon + (\sigma - \tau + \epsilon) = 2\epsilon$, it follows from the applications of [7, 3.642.1], also recalled in the appendix, that

$$C_{0,0,0,[T]} = \frac{1}{a^c b^d}B(c, d).$$

Example 4. Another complex example is now developed. For any $a > 0$, $b > 0$, $c > 0$ and $d > 0$, let us consider

$$T(x, y) = \frac{a\sqrt{x}+b\sqrt{y}}{c\sqrt{x}+d\sqrt{y}}.$$

Then it is clear that $T(x, y)$ is positive and homogeneous of degree 0. Considering the special case $\tau = \sigma$ and $\epsilon = 1$, Theorem 3.1 implies that

$$\int_0^{+\infty}\int_0^{+\infty} \frac{f(x)g(y)}{x+y}\times\frac{a\sqrt{x}+b\sqrt{y}}{c\sqrt{x}+d\sqrt{y}}x^\sigma y^\sigma dxdy \leq C_{1,\sigma,\sigma,[T]}\sqrt{\int_0^{+\infty} x^{2\sigma}f^2(x)dx}\sqrt{\int_0^{+\infty} x^{2\sigma}g^2(x)dx},$$

where

$$C_{1,\sigma,\sigma,[T]} = \int_0^1 \frac{1}{\sqrt{t(1-t)}}\frac{a\sqrt{1-t}+b\sqrt{t}}{c\sqrt{1-t}+d\sqrt{t}}dt.$$

In order to deal with some referenced integral formulas, we propose the change of variables $t = \sin^2(\theta)$. We thus obtain

$$C_{1,\sigma,\sigma,[T]} = 2\int_0^{\pi/2} \frac{a\cos(\theta)+b\sin(\theta)}{c\cos(\theta)+d\sin(\theta)}d\theta.$$

Let us now consider the lemma below.

Lemma 3.2. *Let $a \in \mathbb{R}$, $b \in \mathbb{R}$, $c > 0$ and $d > 0$. Then we have*

$$\int_0^{\pi/2} \frac{a\cos(x)+b\sin(x)}{c\cos(x)+d\sin(x)}dx = \frac{1}{c^2+d^2}\left[(bc-ad)\log\left(\frac{c}{d}\right)+(bd+ac)\frac{\pi}{2}\right].$$

The detailed proof of this lemma is given in the appendix. It follows immediately from this result that

$$C_{1,\sigma,\sigma,[T]} = \frac{2}{c^2+d^2}\left[(bc-ad)\log\left(\frac{c}{d}\right)+(bd+ac)\frac{\pi}{2}\right].$$

Note that $C_{1,\sigma,\sigma,[T]}$ does not depend on σ.

So many more configurations for $T(x,y)$ can be considered, but without necessarily having a closed form expression for $C_{\epsilon,\tau,\sigma,[T]}$.

3.2. Secondary result

In a sense, Theorem 3.1 is capable of self-expansion, with the addition of two more tuning parameters. This claim is developed in the result below.

Proposition 3.3. *Let $\epsilon \geq 0$, $\tau \geq 0$, $\sigma \geq 0$, $\mu > 0$, $\nu > 0$, $f : [0,+\infty) \to [0,+\infty)$ and $g : [0,+\infty) \to [0,+\infty)$ be two functions, and $T : [0,+\infty)^2 \to [0,+\infty)$ be a bivariate function satisfying the following property depending on μ and ν: For any $\lambda > 0$, we have*

$$T(\lambda^{1/\mu}x, \lambda^{1/\nu}y) = T(x,y).$$

Then we have

$$\int_0^{+\infty}\int_0^{+\infty}\frac{f(x)g(y)}{(x^\mu+y^\nu)^\epsilon}T(x,y)x^\tau y^\sigma dxdy$$
$$\leq D_{\epsilon,\tau,\sigma,\mu,\nu,[T]}\sqrt{\int_0^{+\infty}x^{\tau+(\sigma+1)\mu/\nu-\mu\epsilon}f^2(x)dx}\sqrt{\int_0^{+\infty}x^{(\tau+1)\nu/\mu+\sigma-\nu\epsilon}g^2(x)dx},$$

where

$$D_{\epsilon,\tau,\sigma,\mu,\nu,[T]} = \frac{1}{\sqrt{\mu\nu}}\int_0^1(1-t)^{[(\tau+1)/\mu-(\sigma+1)/\nu+\epsilon]/2-1}t^{[(\sigma+1)/\nu-(\tau+1)/\mu+\epsilon]/2-1}\times$$
$$T[(1-t)^{1/\mu},t^{1/\nu}]dt,$$

provided that the integrals on the right-hand side of the inequality exist.

Proof. For the first step, by the change of variables $p = x^\mu$ and $q = y^\nu$, we have

$$\int_0^{+\infty}\int_0^{+\infty}\frac{f(x)g(y)}{(x^\mu+y^\nu)^\epsilon}T(x,y)x^\tau y^\sigma dxdy$$
$$= \frac{1}{\mu\nu}\int_0^{+\infty}\int_0^{+\infty}\frac{f(p^{1/\mu})g(q^{1/\nu})}{(p+q)^\epsilon}T(p^{1/\mu},q^{1/\nu})p^{(\tau+1)/\mu-1}q^{(\sigma+1)/\nu-1}dpdq$$
$$= \frac{1}{\mu\nu}\int_0^{+\infty}\int_0^{+\infty}\frac{f_*(p)g_*(q)}{(p+q)^\epsilon}T_*(p,q)p^{\tau_*}q^{\sigma_*}dpdq, \tag{3.3}$$

where

$$f_*(p) = f(p^{1/\mu}), \quad g_*(q) = g(q^{1/\nu}), \quad T_*(p,q) = T(p^{1/\mu},q^{1/\nu}),$$

and

$$\tau_* = \frac{1}{\mu}(\tau+1)-1, \quad \sigma_* = \frac{1}{\nu}(\sigma+1)-1.$$

Thanks to the assumption made on $T(x, y)$, for any $\lambda > 0$, we have

$$T_*(\lambda p, \lambda q) = T\left[(\lambda p)^{1/\mu}, (\lambda q)^{1/\nu}\right] = T\left[\lambda^{1/\mu} p^{1/\mu}, \lambda^{1/\nu} q^{1/\nu}\right] \\ = T\left[p^{1/\mu}, q^{1/\nu}\right] = T_*(p, q).$$

This implies that $T_*(p, q)$ is homogeneous of degree 0. Therefore, we can apply Theorem 3.1 under this exact configuration, which gives

$$\int_0^{+\infty}\int_0^{+\infty} \frac{f_*(p)g_*(q)}{(p+q)^{\epsilon}} T_*(p, q) p^{\tau_*} q^{\sigma_*} dpdq \\ \leq C_{\epsilon,\tau_*,\sigma_*,[T_*]}\sqrt{\int_0^{+\infty} p^{\tau_*+\sigma_*-\epsilon+1} f_*^2(p)dp}\sqrt{\int_0^{+\infty} q^{\tau_*+\sigma_*-\epsilon+1} g_*^2(q)dq}, \tag{3.4}$$

with

$$C_{\epsilon,\tau_*,\sigma_*,[T_*]} = \int_0^1 (1-t)^{(\tau_*-\sigma_*+\epsilon)/2-1} t^{(\sigma_*-\tau_*+\epsilon)/2-1} T_*(1-t, t)dt \\ = \int_0^1 (1-t)^{[(\tau+1)/\mu-(\sigma+1)/\nu+\epsilon]/2-1} t^{[(\sigma+1)/\nu-(\tau+1)/\mu+\epsilon]/2-1} T[(1-t)^{1/\mu}, t^{1/\nu}]dt \\ = \sqrt{\mu\nu} D_{\epsilon,\tau,\sigma,\mu,\nu,[T]}.$$

Considering the changes of variables $p = x^{\mu}$ and $q = x^{\nu}$ independently, we have

$$\sqrt{\int_0^{+\infty} p^{\tau_*+\sigma_*-\epsilon+1} f_*^2(p)dp}\sqrt{\int_0^{+\infty} q^{\tau_*+\sigma_*-\epsilon+1} g_*^2(q)dq} \\ = \sqrt{\int_0^{+\infty} p^{(\tau+1)/\mu+(\sigma+1)/\nu-\epsilon-1} f^2(p^{1/\mu})dp} \times \\ \sqrt{\int_0^{+\infty} q^{(\tau+1)/\mu+(\sigma+1)/\nu-\epsilon-1} g^2(q^{1/\nu})dq} \\ = \sqrt{\mu\nu}\sqrt{\int_0^{+\infty} x^{\tau+(\sigma+1)\mu/\nu-\mu\epsilon} f^2(x)dx}\sqrt{\int_0^{+\infty} x^{(\tau+1)\nu/\mu+\sigma-\nu\epsilon} g^2(x)dx}. \tag{3.5}$$

If we combine Equations (3.3), (3.4) and (3.5), we obtain

$$\int_0^{+\infty}\int_0^{+\infty} \frac{f(x)g(y)}{(x^{\mu}+y^{\nu})^{\epsilon}} T(x, y) x^{\tau} y^{\sigma} dxdy \\ \leq \frac{1}{\mu\nu}\sqrt{\mu\nu}\sqrt{\mu\nu} D_{\epsilon,\tau,\sigma,\mu,\nu,[T]}\sqrt{\int_0^{+\infty} x^{\tau+(\sigma+1)\mu/\nu-\mu\epsilon} f^2(x)dx} \times \\ \sqrt{\int_0^{+\infty} x^{(\tau+1)\nu/\mu+\sigma-\nu\epsilon} g^2(x)dx} \\ = D_{\epsilon,\tau,\sigma,\mu,\nu,[T]}\sqrt{\int_0^{+\infty} x^{\tau+(\sigma+1)\mu/\nu-\mu\epsilon} f^2(x)dx}\sqrt{\int_0^{+\infty} x^{(\tau+1)\nu/\mu+\sigma-\nu\epsilon} g^2(x)dx}.$$

This ends the proof of Proposition 3.3. ∎

Clearly, by selecting $\mu = 1$ and $\nu = 1$, Proposition 3.3 becomes Theorem 3.1. Taking $T(x, y) = 1$, we get

$$\int_0^{+\infty} \int_0^{+\infty} \frac{f(x)g(y)}{(x^\mu + y^\nu)^\epsilon} x^\tau y^\sigma dx dy \leq D_{\epsilon,\tau,\sigma,\mu,\nu,[T]} \sqrt{\int_0^{+\infty} x^{\tau+(\sigma+1)\mu/\nu-\mu\epsilon} f^2(x) dx} \sqrt{\int_0^{+\infty} x^{(\tau+1)\nu/\mu+\sigma-\nu\epsilon} g^2(x) dx},$$

where

$$D_{\epsilon,\tau,\sigma,\mu,\nu,[T]} = \frac{1}{\sqrt{\mu\nu}} \int_0^1 (1-t)^{[(\tau+1)/\mu-(\sigma+1)/\nu+\epsilon]/2-1} t^{[(\sigma+1)/\nu-(\tau+1)/\mu+\epsilon]/2-1} dt = \frac{1}{\sqrt{\mu\nu}} B\left\{ \frac{1}{2}\left[\frac{1}{\mu}(\tau+1) - \frac{1}{\nu}(\sigma+1) + \epsilon\right], \frac{1}{2}\left[\frac{1}{\nu}(\sigma+1) - \frac{1}{\mu}(\tau+1) + \epsilon\right] \right\}.$$

So many more results of this kind can be determined with more complex choices for $T(x, y)$. This, in addition to the use of multiple tuning parameters, shows the generality of Proposition 3.3 and Theorem 2.1.

4. Conclusion

In conclusion, this paper reintroduces and extends the approach of David C. Ullrich in 2013 by deriving two new generalized Hilbert integral inequalities. They are innovative in the following sense: the first one considers an original domain of integration, a triangle depending on a tuning parameter, and the second one includes several tuning parameters and a general homogeneous kernel of degree 0. The results presented thus highlight the versatility, relative simplicity and elegance of this approach. They also bring new insights to the field, with specific examples illustrating its potential for further exploration.

In particular, one possible perspective is to show that the "parametric functions" obtained in the upper bounds are optimal in the mathematical sense. One might also consider investigating the upper bound for the following variant of the double integral in Theorem 2.1:

$$\int_0^\beta \left[\int_0^{\beta - x^\delta} \frac{f(x)g(y)}{(x+y)^\epsilon} dy \right] dx,$$

where $\delta > 0$ is a new additional parameter, as well as some kinds of multidimensional versions. We will leave these possibilities for future work.

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References

[1] J. Aczel, *Lectures on Functional Equations and Their Applications Mathematics in Science and Engineering*, Vol. 19, Academic Press, New York, 1966.

[2] V. Adiyasuren, T. Batbold, M. Krnić, Hilbert-type inequalities involving differential operators, the best constants and applications, *Math. Inequal. Appl.* **18** (2015) 111–124.

[3] V. Adiyasuren, T. Batbold, M. Krnić, Multiple Hilbert-type inequalities involving some differential operators, *Banach J. Math. Anal.* **10** (2016) 320–337.

[4] L. E. Azar, The connection between Hilbert and Hardy inequalities, *J. Inequal. Appl.* **2013** (2013) 1–10.

[5] T. Batbold, Y. Sawano, Sharp bounds for m-linear Hilbert-type operators on the weighted Morrey spaces, *Math. Inequal. Appl.* **20** (2017) 263–283.

[6] Q. Chen, B. C. Yang, A survey on the study of Hilbert-type inequalities, *J. Inequal. Appl.* **2015** (2015) 1–30.

[7] I. S. Gradshteyn, I. M. Ryzhik, *Table of Integrals, Series, and Products*, 7th Edition, Academic Press, Burlington, 2007.

[8] G. H. Hardy, J. E. Littlewood, G. Polya, *Inequalities*, Cambridge University Press, Cambridge, 1934.

[9] M. H. Protter, C. B. Jr. Morrey, *Intermediate Calculus*, Springer, New York, 1985.

[10] D. C. Ullrich, A simple elementary proof of Hilbert's inequality, *Amer. Math. Monthly* **120** (2013) 161–164.

[11] Z. T. Xie, Z. Zeng, Y. F. Sun, A new Hilbert-type inequality with the homogeneous kernel of degree -2, *Adv. Appl. Math. Sci.* **12** (2013) 391–401.

[12] D. M. Xin, A Hilbert-type integral inequality with the homogeneous kernel of zero degree, *Math. Theory Appl.* **30** (2010) 70–74.

[13] J. S. Xu, Hardy-Hilbert's inequalities with two parameters, *Adv. Math.* **36** (2007) 63–76.

[14] B. C. Yang, On Hilbert's integral inequality, *J. Math. Anal. Appl.* **220** (1998) 778–785.

[15] B. C. Yang, *The Norm of Operator and Hilbert-Type Inequalities*, Science Press, Beijing, 2009.

[16] B. C. Yang, *Hilbert-Type Integral Inequalities*, Bentham Science Publishers, The United Arab Emirates, 2009.

[17] B. C. Yang, On the norm of an integral operator and applications, *J. Math. Anal. Appl.* **321** (2006) 182–192.

[18] B. C. Yang, On the norm of a Hilbert's type linear operator and applications, *J. Math. Anal. Appl.* **325** (2007) 529–541.

[19] B. C. Yang, A note on Hilbert's integral inequalities, *Chin. Q. J. Math.* **13** (1998) 83–86.

[20] Z. Zhen, K. Raja Rama Gandhi, Z. T. Xie, A new Hilbert-type inequality with the homogeneous kernel of degree -2 and with the integral in whole plane, *Bull. Math. Sci. Appl.* **3** (2014) 11–20.

Appendix

- **Fubini-Tonelli integral theorem.** Let $(X, \mathcal{A}, \phi)$ and $(Y, \mathcal{B}, \psi)$ be σ-finite measure spaces and $f : X \times Y \mapsto [0, +\infty)$ be a measurable function. Then the following equalities hold:
$$\int_{X\times Y} f(x,y)d(\phi\times\psi)(x,y) = \int_X \left[\int_Y f(x,y)d\psi(y)\right] d\phi(x)$$
$$= \int_Y \left[\int_X f(x,y)d\phi(x)\right] d\psi(y).$$
Note that the exchange of the order of integration holds true even if the integrals are divergent.

- **Exact statement of [7, 3.642.1].** Let $\mu > 0$ and $\nu > 0$. Then we have
$$\int_0^{\pi/2} \frac{\sin^{2\mu-1}(x)\cos^{2\nu-1}(x)}{[a^2\sin^2(x)+b^2\cos^2(x)]^{\mu+\nu}}dx = \frac{1}{2a^{2\mu}b^{2\nu}}B(\mu,\nu).$$

- **Proof of Lemma 3.2.** Applying the change of variables $t = \tan(x)$, so $dt = [1+\tan^2(x)]dx = (1+t^2)dx$, we get
$$\int_0^{\pi/2} \frac{a\cos(x)+b\sin(x)}{c\cos(x)+d\sin(x)}dx = \int_0^{\pi/2} \frac{a+b\tan(x)}{c+d\tan(x)}dx = \int_0^{+\infty} \frac{a+bt}{(c+dt)(1+t^2)}dt.$$
A fractional decomposition in simple elements gives
$$\frac{a+bt}{(c+dt)(1+t^2)} = \frac{U}{c+dt} + \frac{Vt+W}{1+t^2},$$
where
$$U = \frac{d(ad-bc)}{c^2+d^2}, \quad V = \frac{bc-ad}{c^2+d^2}, \quad W = \frac{ac+bd}{c^2+d^2}.$$
Using appropriate primitives and noticing that $U = -dV$, we obtain
$$\int_0^{+\infty} \frac{a+bt}{(c+dt)(1+t^2)}dt = \left[\frac{U}{d}\log(c+dt) + \frac{V}{2}\log(1+t^2) + W\arctan(t)\right]_{t=0}^{t\to+\infty}$$
$$= \left\{V\log\left[\frac{\sqrt{1+t^2}}{c+dt}\right] + W\arctan(t)\right\}_{t=0}^{t\to+\infty}$$
$$= -V\log(d) + W\frac{\pi}{2} - [-V\log(c)] = V\log\left(\frac{c}{d}\right) + W\frac{\pi}{2}$$
$$= \frac{1}{c^2+d^2}\left[(bc-ad)\log\left(\frac{c}{d}\right) + (bd+ac)\frac{\pi}{2}\right].$$
This concludes the proof of Lemma 3.2.