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A new type of convergence in partial metric spaces

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Abstract In this paper, we introduce the concept of deferred statistical convergence in partial metric spaces (pms), extending classical notions of statistical convergence and summability. We define deferred Cesàro summability and investigate its fundamental properties. Connections between statistical convergence and deferred Cesàro summability are explored, including inclusion relationships and strictness. Additionally, we establish conditions under which deferred summability implies statistical convergence and vice versa. Examples and theorems are provided to illustrate the applicability and relevance of these concepts in partial metric spaces.

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1. Introduction

Statistical convergence extends the classical notion of convergence by incorporating the density of index sets. Given a subset $S \subseteq \mathbb{N}$, its density $\delta(S)$ is given by

$$\delta(S) = \lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} \mathbb{I}_S(j),$$

whenever the limit exists, where $\mathbb{I}_S$ denotes the indicator function of S, defined as

$$\mathbb{I}_S(j) = \begin{cases} 1, & j \in S, \\ 0, & j \notin S. \end{cases}$$

This density-based approach allows for a more flexible analysis of sequence behavior, particularly in cases where traditional pointwise convergence fails. The concept of statistical convergence was independently introduced by Steinhaus [13] and Fast [4] in the early

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1950s. A sequence $\{\xi_n\}$ of real or complex numbers is said to be statistically convergent to ξ, denoted as $\text{st-}\lim \xi_n = \xi$, if for every $\varepsilon > 0$,

$$\delta(\{k \in \mathbb{N} : |\xi_k - \xi| \geq \varepsilon\}) = 0.$$

Equivalently, there exists a subset $S \subseteq \mathbb{N}$ with $\delta(S) = 1$ such that for all sufficiently large indices $k \in S$, we have $|\xi_k - \xi| < \varepsilon$; see [5]. This definition extends classical convergence by allowing a sequence to converge "almost everywhere" in terms of density rather than requiring convergence for all indices beyond some fixed threshold.

Furthermore, a sequence $\{\xi_n\}$ is called statistically Cauchy if, for each $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$\delta(\{k \in \mathbb{N} : |\xi_k - \xi_N| \geq \varepsilon\}) = 0.$$

This condition ensures that, with density one, the terms of the sequence become arbitrarily close to each other as the index grows.

In recent decades, statistical convergence has gained significant importance in various branches of mathematics, including summability theory, number theory, probability, measure theory, optimization, and approximation theory. Its flexibility in handling irregular sequences has led to numerous extensions and applications across different mathematical fields.

Deferred statistical convergence, a more recent generalization, introduces two sequences $\{\nu(n)\}$ and $\{\omega(n)\}$, which determine intervals for assessing convergence. This refinement has proven useful for sequences with non-uniform structures, offering new insights into summability theory and metric spaces.

In [1], Agnew introduced the concept of the *deferred Cesàro mean* for sequences of real or complex numbers $\{\xi_k\}$. This mean is defined as

$$(D_{\nu,\omega}x)_n = \frac{1}{\omega(n) - \nu(n)} \sum_{k=\nu(n)+1}^{\omega(n)} \xi_k, \quad n = 1, 2, 3, \ldots,$$

where $\nu = \{\nu(n)\}$ and $\omega = \{\omega(n)\}$ are sequences of non-negative integers satisfying

$$\nu(n) < \omega(n) \quad \text{and} \quad \lim_{n\to\infty} \omega(n) = \infty.$$

Unlike the classical Cesàro mean, which averages terms starting from the first element of the sequence, the deferred Cesàro mean operates over a more flexible interval $[\nu(n) + 1, \omega(n)]$. By adjusting the selection of terms included in the averaging process, this approach extends the applicability of Cesàro-type summability methods, offering greater versatility in the study of sequence convergence and related topics.

Partial metric spaces (pms) (S, d), first introduced by S. Matthews in 1994, provide a natural extension of classical metric spaces by relaxing the condition $d(a, a) = 0$ for all $a \in S$. In a pms (S, d), the self-distance $d(a, a)$ may be nonzero, reflecting a broader and more flexible framework for distance measurement. This generalization has enabled pms to find diverse applications across various fields, including topology, computer science, information theory, and biological sciences [8].

A key result in fixed-point theory, the Banach contraction principle, originally developed for complete metric spaces, has been extended to pms. These extensions often involve generalizing the class of mappings, broadening the domain, or combining both approaches, demonstrating the versatility of pms in theoretical and applied mathematics.

This paper defines deferred statistical convergence in pms and investigates its foundational properties. We also provide examples and examine their connections to existing notions of convergence.

This paper is built upon the concepts and results presented in the foundation works [1]-[14]. Interested readers may refer to these papers for additional background and comprehensive discussions on the topics of statistical convergence, deferred Cesàro means, and pms.

2. Preliminaries

Definition 2.1. A partial metric space (pms) consists of a pair (S, d), where S is a nonempty set, and $d : S \times S \to \mathbb{R}^+$ is a function satisfying the following conditions for all $a, b, c \in S$.

(1) $d(a, a) \leq d(a, b)$ (self-referential property),

(2) $d(a, b) = d(b, a)$ (symmetry condition),

(3) If $d(a, a) = d(b, b) = d(a, b)$, then $a = b$ (identity property),

(4) $d(a, b) \leq d(a, c) + d(c, b) - d(c, c)$ (generalized triangle inequality).

This framework generalizes classical metric spaces by allowing self-distances $d(a, a)$ to be nonzero, making it particularly useful in domain theory and computational models.

The topology induced by a partial metric d is generated by modified open balls, see [8], $B_d(x, \varepsilon) = \{y \in S : d(x, y) < d(x, x) + \varepsilon\}$. Every metric space is also a partial metric space, but not every partial metric space is a metric space. Let p be a partial metric on S. Define $p_s : S \times S \to [0, \infty)$ by

$$p_s(x, y) = 2p(x, y) - p(x, x) - p(y, y).$$

Then p_s is a (classical) metric on S.

Example 2.2. Let $S = [0, \infty)$ and define a partial metric $p : S \times S \to [0, \infty)$ by

$$p(x, y) = \max\{x, y\}.$$

Consider the sequence (x_n) defined by

$$x_n = \begin{cases} 0, & \text{if } n \text{ is even}, \\ 1, & \text{if } n \text{ is odd}, \end{cases} \quad \text{and let } x = 1.$$

(i) *In the classical metric* $d(x, y) = |x - y|$: The sequence alternates between 0 and 1, so we compute

$$|x_n - 1| = \begin{cases} 1, & \text{if } n \text{ is even}, \\ 0, & \text{if } n \text{ is odd}. \end{cases}$$

Thus, the sequence does not converge to $x = 1$ in the usual metric, as it does not have a limit in the standard sense.

(ii) *In the partial metric* $p(x, y) = \max\{x, y\}$: We compute

$$p(x_n, 1) = \max\{x_n, 1\} = 1 \quad \text{for all } n,$$

and

$$p(1, 1) = \max\{1, 1\} = 1.$$

Hence,

$$\lim_{n\to\infty} p(x_n,1) = 1 = p(1,1),$$

which means $x_n \to 1$ in the partial metric space (S,p). This example shows that a sequence may converge in a partial metric space without converging in the classical metric. It highlights a fundamental distinction between the two notions of convergence.

In the context of sequence analysis, the concept of *deferred statistical convergence* (D-statistical convergence) involves two integer sequences, $\{\nu(n)\}$ and $\{\omega(n)\}$, satisfying $\nu(n) < \omega(n)$ and $\omega(n) \to \infty$. A sequence $\{\xi_k\}$ is said to be *D-statistically convergent* to L if for any $\varepsilon > 0$, see [7],

$$\lim_{n\to\infty} \frac{1}{\omega(n)-\nu(n)} \left|\{k \mid \nu(n) < k \le \omega(n), |\xi_k - L| \ge \varepsilon\}\right| = 0.$$

Here, the notation $|\cdot|$ denotes the cardinality of the enclosed set.

In the setting of pms (S,d), a sequence $\{\xi_k\}$ is said to be *statistically convergent* to $a \in S$ if, see [10],

$$\delta(\{k \in \mathbb{N} : |d(\xi_k,a) - d(a,a)| \ge \varepsilon\}) = 0, \quad \forall \varepsilon > 0.$$

This formulation extends statistical convergence by incorporating the structural flexibility of pms.

3. D-Statistical Convergence and D-Cesàro Summability

In this section, we define the notions of D-statistical convergence and D-strong Cesàro summability within the framework of pms.

Definition 3.1. Let (S,d) be a pms, and consider two sequences of nonnegative integers, $\{\nu(n)\}$ and $\{\omega(n)\}$, such that $\nu(n) < \omega(n)$ and $\omega(n) \to \infty$. A sequence $\{\xi_k\}$ in S is said to be *D-statistically convergent* to $a \in S$ if for any $\varepsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{\omega(n)-\nu(n)} \left|\{k \mid \nu(n) < k \le \omega(n), d(\xi_k,a) - d(a,a) \ge \varepsilon\}\right| = 0.$$

Moreover, we write $\mathrm{DS}_{d,\nu,\omega}\text{-}\lim \xi_k = a$ or $\xi_k \to a$ $(\mathrm{DS}_{d,\nu,\omega})$ whenever $\{\xi_k\}$ is D-statistically convergent to a.

Example 3.2. Consider the space $S = \mathbb{R}$ with the function $d(x,y) = \max\{x,y\}$. Let the sequence be defined as $\xi_k = \frac{1}{k}$, with $\nu(n) = n-1$ and $\omega(n) = n+1$. Then, it follows that $\{\xi_k\}$ is D-statistically convergent to 0 as $n \to \infty$.

Definition 3.3. Given a pms (S,d) and sequences $\{\nu(n)\}$ and $\{\omega(n)\}$ of nonnegative integers such that $\nu(n) < \omega(n)$ and $\omega(n) \to \infty$, a sequence $\{\xi_k\}$ in S is said to be *D-strongly Cesàro summable* to $\xi \in S$ if

$$\lim_{n\to\infty} \frac{1}{\omega(n)-\nu(n)} \sum_{k=\nu(n)+1}^{\omega(n)} |d(\xi_k,\xi) - d(\xi,\xi)| = 0.$$

This summability is represented by

$$\mathrm{Dwd}_{d,\nu,\omega}\text{-}\lim \xi_k = \xi \quad \text{or} \quad \xi_k \to \xi \ (\mathrm{Dwd}_{d,\nu,\omega}).$$

The set of all such sequences is denoted as $\mathrm{Dwd}_{d,\nu,\omega}$.

If the sequences $\{\omega(n)\}$ and $\{\nu(n)\}$ are chosen such that $\omega(n) = n$ and $\nu(n) = 0$ for all $n \in \mathbb{N}$, the concept of D-strong Cesàro summability coincides with the classical strong Cesàro summability, since the deferred averaging process aligns with the standard Cesàro mean.

Example 3.4. Consider the pms (S, d), where $S = [0, \infty)$ and the partial metric is defined as $d(x, y) = \max\{x, y\}$. Define the sequence $\{\xi_k\}$ as follows.

$$\xi_k = \begin{cases} \frac{1}{k}, & \text{if } k \text{ is a perfect square,} \\ 1, & \text{otherwise.} \end{cases}$$

We will demonstrate that the sequence $\{\xi_k\}$ is not D-convergent but is D-statistically convergent to 1 using $\nu(n) = n$ and $\omega(n) = 2n$. To check whether $\{\xi_k\}$ is D-convergent to 1, we need to verify the condition

$$\lim_{n\to\infty} \frac{1}{\omega(n) - \nu(n)} \sum_{k=\nu(n)+1}^{\omega(n)} |d(\xi_k, 1) - d(1, 1)| = 0.$$

Here, $d(\xi_k, 1) = \max\{\xi_k, 1\}$, and $d(1, 1) = 1$. For perfect squares $k = m^2$, $\xi_k = \frac{1}{k}$. Therefore,

$$|d(\xi_k, 1) - d(1, 1)| = \begin{cases} 1 - 1 = 0, & \text{if } \xi_k \leq 1, \\ \frac{1}{k} - 1, & \text{if } \xi_k > 1. \end{cases}$$

For non-perfect squares, $\xi_k = 1$, so $|d(\xi_k, 1) - d(1, 1)| = 0$. The contribution of perfect squares prevents the sum from approaching 0, hence $\{\xi_k\}$ is not D-convergent.

Now, we verify D-statistical convergence to 1. Define the set

$$A_{n,\varepsilon} = \{k : \nu(n) < k \leq \omega(n), |d(\xi_k, 1) - d(1, 1)| \geq \varepsilon\}.$$

This set includes only perfect squares $k = m^2$ such that

$$|d(\xi_k, 1) - d(1, 1)| = \frac{1}{k} - 1 \geq \varepsilon \implies k \leq \frac{1}{1 + \varepsilon}.$$

For $\nu(n) = n$ and $\omega(n) = 2n$, the number of perfect squares in the interval $\nu(n) < k \leq \omega(n)$ is

$$|A_{n,\varepsilon}| = \lfloor\sqrt{2n}\rfloor - \lfloor\sqrt{n}\rfloor.$$

The D-statistical density is then

$$\frac{|A_{n,\varepsilon}|}{\omega(n) - \nu(n)} = \frac{\lfloor\sqrt{2n}\rfloor - \lfloor\sqrt{n}\rfloor}{n}.$$

Simplifying this expression, we have

$$\frac{\sqrt{2n} - \sqrt{n}}{n} = \frac{\sqrt{n}(\sqrt{2} - 1)}{n} = \frac{\sqrt{2} - 1}{\sqrt{n}}.$$

As $n \to \infty$, the density approaches 0:

$$\lim_{n\to\infty} \frac{|A_{n,\varepsilon}|}{\omega(n) - \nu(n)} = 0.$$

This proves that $\{\xi_k\}$ is D-statistically convergent to 1.

Theorem 3.5. *Let (S, d) be a pms, and let $\{\xi_k\}$, $\{\alpha_k\}$ be sequences in S. Then* *(i) If* $DS_{d,\nu,\omega}\text{-}\lim \xi_k = \xi_0$ *and* $DS_{d,\nu,\omega}\text{-}\lim \alpha_k = \alpha_0$*, then*

$$DS_{d,\nu,\omega}\text{-}\lim(\xi_k + \alpha_k) = \xi_0 + \alpha_0,$$

provided that addition is defined in S*.*

(ii) If $DS_{d,\nu,\omega}\text{-}\lim \xi_k = \xi_0$ *and* $c \in \mathbb{C}$*, then*

$$DS_{d,\nu,\omega}\text{-}\lim(c\xi_k) = c\xi_0,$$

provided that scalar multiplication is defined in S*.*

(iii) If $DS_{d,\nu,\omega}\text{-}\lim \xi_k = \xi_0$*,* $DS_{d,\nu,\omega}\text{-}\lim \alpha_k = \alpha_0$*, and* $\{\xi_k\}, \{\alpha_k\} \in \ell_\infty$*, then*

$$DS_{d,\nu,\omega}\text{-}\lim(\xi_k \alpha_k) = \xi_0 \alpha_0,$$

where ℓ_∞ *is the space of bounded sequences in* S*.*

Proof. (i) Given $\mathrm{DS}_{d,\nu,\omega}\text{-}\lim \xi_k = \xi_0$ and $\mathrm{DS}_{d,\nu,\omega}\text{-}\lim \alpha_k = \alpha_0$, we will show that

$$\mathrm{DS}_{d,\nu,\omega}\text{-}\lim(\xi_k + \alpha_k) = \xi_0 + \alpha_0.$$

By the properties of the partial metric,

$$d(\xi_k + \alpha_k, \xi_0 + \alpha_0) \leq d(\xi_k, \xi_0) + d(\alpha_k, \alpha_0).$$

Since both sequences are D-statistically convergent, the density of indices where $d(\xi_k, \xi_0) + d(\alpha_k, \alpha_0) \geq \varepsilon$ vanishes as $n \to \infty$, proving the claim.

(ii) For $\mathrm{DS}_{d,\nu,\omega}\text{-}\lim \xi_k = \xi_0$ and scalar $c \in \mathbb{C}$, we verify $\mathrm{DS}_{d,\nu,\omega}\text{-}\lim(c\xi_k) = c\xi_0$. Using the metric property,

$$d(c\xi_k, c\xi_0) = |c| d(\xi_k, \xi_0),$$

we see that the density of indices where $d(c\xi_k, c\xi_0) \geq \varepsilon$ vanishes as $n \to \infty$, ensuring convergence.

(iii) Given $\mathrm{DS}_{d,\nu,\omega}\text{-}\lim \xi_k = \xi_0$, $\mathrm{DS}_{d,\nu,\omega}\text{-}\lim \alpha_k = \alpha_0$, and boundedness $\xi_k, \alpha_k \in \ell_\infty$, we show $\mathrm{DS}_{d,\nu,\omega}\text{-}\lim(\xi_k \alpha_k) = \xi_0 \alpha_0$. The estimate

$$d(\xi_k \alpha_k, \xi_0 \alpha_0) \leq d(\xi_k, \xi_0)|\alpha_k| + d(\alpha_k, \alpha_0)|\xi_k|$$

with bounded ξ_k, α_k implies the density of indices where $d(\xi_k \alpha_k, \xi_0 \alpha_0) \geq \varepsilon$ tends to zero, proving the result. ■

The proofs of the following theorems utilize techniques similar to those employed in [7],[2],[10], and [3]. These foundational works provided inspiration and methodological guidance for our results. Readers interested in the detailed methodologies are encouraged to refer to these references for further insights.

Theorem 3.6. *Let* (S, d) *be a pms. Then*

$$Dwd_{d,\nu,\omega} \subseteq DS_{d,\nu,\omega},$$

and the inclusion is strict.

Proof. The first part of the proof follows directly and is omitted. To establish the strictness of the inclusion, consider the sequences $\omega(n) = n$ and $\nu(n) = 0$ for all $n \in \mathbb{N}$, and set $a = 0$. Define the sequence $\{\xi_k\}$ in X by:

$$\xi_k = \begin{cases} \frac{\sqrt{n}}{2}, & \text{if } k = n^2, \\ \\ 0, & \text{otherwise.} \end{cases}$$

For any $\varepsilon > 0$, we examine

$$\frac{1}{\omega(n) - \nu(n)} \left| \{k \in (\nu(n), \omega(n)] : |d(\xi_k, 0) - d(0,0)| \geq \varepsilon\} \right|.$$

Since $|d(\xi_k, 0) - d(0,0)| = \frac{\sqrt{n}}{2}$ only for indices $k = n^2$, the proportion of such indices is given by

$$\frac{1}{\omega(n) - \nu(n)} |\{k \leq n : k = n^2\}| = \frac{\sqrt{n}}{n} \to 0 \quad \text{as } n \to \infty.$$

Thus, ξ_k is D-statistically convergent to 0, i.e., $\xi_k \to 0$ in the sense of $\mathrm{DS}_{d,\nu,\omega}$.

Next, we consider the Cesàro mean

$$\frac{1}{\omega(n) - \nu(n)} \sum_{k=\nu(n)+1}^{\omega(n)} |d(\xi_k, 0) - d(0,0)|.$$

Since $|d(\xi_k, 0) - d(0,0)| = \frac{\sqrt{n}}{2}$ for $k = n^2$, the sum evaluates to

$$\frac{1}{\omega(n) - \nu(n)} \sum_{m=1}^{\sqrt{n}} \frac{\sqrt{n}}{2} = \frac{\sqrt{n}}{2} \cdot \frac{\sqrt{n}}{n} = \frac{n}{2n} = \frac{1}{2}.$$

Since this does not tend to zero as $n \to \infty$, it follows that ξ_k is not strongly D-Cesàro summable to 0, meaning $\xi_k \notin \mathrm{Dwd}_{d,\nu,\omega}$.

This demonstrates that $\mathrm{Dwd}_{d,\nu,\omega} \subseteq \mathrm{DS}_{d,\nu,\omega}$, but the inclusion is strict. ∎

Theorem 3.7. *If* $\liminf_{n\to\infty} \frac{\omega(n)}{\nu(n)} > 1$, *then* $S_d \subseteq DS_{d,\nu,\omega}$.

Proof. Assume that $\liminf_{n\to\infty} \frac{\omega(n)}{\nu(n)} > 1$; we can find a $\eta > 0$ such that $\frac{\omega(n)}{\nu(n)} \geq 1 + \eta$ for sufficiently large n. This implies

$$\frac{\omega(n) - \nu(n)}{\omega(n)} \geq \frac{\eta}{1+\eta} \implies \frac{1}{\omega(n)} \geq \frac{\eta}{(1+\eta)(\omega(n) - \nu(n))}.$$

If $\xi_k \to a$ in the sense of S_d, then for every $\varepsilon > 0$ and sufficiently large n, we have

$$\begin{aligned} &\frac{1}{\omega(n)} \left| \{k \leq \omega(n) : |d(\xi_k, a) - d(a,a)| \geq \varepsilon\} \right| \\ &\quad \geq \frac{1}{\omega(n)} \left| \{\nu(n) < k \leq \omega(n) : |d(\xi_k, a) - d(a,a)| \geq \varepsilon\} \right|. \end{aligned}$$

Using the above inequality, we obtain

$$\begin{aligned} &\frac{1}{\omega(n)} \left| \{\nu(n) < k \leq \omega(n) : |d(\xi_k, a) - d(a,a)| \geq \varepsilon\} \right| \\ &\quad \geq \frac{\eta}{1+\eta} \frac{1}{\omega(n) - \nu(n)} \left| \{\nu(n) < k \leq \omega(n) : |d(\xi_k, a) - d(a,a)| \geq \varepsilon\} \right|. \end{aligned}$$

Thus, $\mathrm{S}_d \subseteq \mathrm{DS}_{d,\nu,\omega}$. ∎

Theorem 3.8. *If* $\lim_{n\to\infty} \inf \frac{\omega(n)-\nu(n)}{n} > 0$ *and* $\omega(n) < n$, *then* $S_d \subseteq DS_{d,\nu,\omega}$.

Proof. Let $\lim_{n\to\infty}\inf \frac{\omega(n)-\nu(n)}{n} > 0$ and $\omega(n) < n$. Then, for every $\varepsilon > 0$, the inclusion

$$\{k \le n : |d(\xi_k, a) - d(a,a)| \ge \varepsilon\} \supseteq \{\nu(n) < k \le \omega(n) : |d(\xi_k, a) - d(a,a)| \ge \varepsilon\}$$

is satisfied. Using this, we have

$$\begin{aligned}&\frac{1}{n}\left|\{k \le n : |d(\xi_k, a) - d(a,a)| \ge \varepsilon\}\right| \\ &\quad \ge \frac{1}{n}\left|\{\nu(n) < k \le \omega(n) : |d(\xi_k, a) - d(a,a)| \ge \varepsilon\}\right|.\end{aligned}$$

Simplifying further,

$$\begin{aligned}&\frac{1}{n}\left|\{\nu(n) < k \le \omega(n) : |d(\xi_k, a) - d(a,a)| \ge \varepsilon\}\right| \\ &\quad = \frac{\omega(n)-\nu(n)}{n}\cdot\frac{1}{\omega(n)-\nu(n)}\left|\{\nu(n) < k \le \omega(n) : |d(\xi_k, a) - d(a,a)| \ge \varepsilon\}\right|.\end{aligned}$$

Thus, $\mathrm{S}_d \subseteq \mathrm{DS}_{d,\nu,\omega}$. ∎

Theorem 3.9. *Let $\{\nu(n)\}$, $\{\omega(n)\}$, $\{\nu_0(n)\}$, and $\{\omega_0(n)\}$ be four sequences of non-negative integers such that*

$$\nu_0(n) < \nu(n) < \omega(n) < \omega_0(n) \quad \textit{for all } n \in \mathbb{N}.$$

Then,

(i) If $\lim_{n\to\infty}[\omega(n)-\nu(n)][\omega_0(n)-\nu_0(n)]^{-1} = \mu > 0$, then $DS_{d,\nu_0,\omega_0} \subseteq DS_{d,\nu,\omega}$.

(ii) If $\lim_{n\to\infty}[\omega_0(n)-\nu_0(n)][\omega(n)-\nu(n)]^{-1} = 1$, then $DS_{d,\nu,\omega} \subseteq DS_{d,\nu_0,\omega_0}$.

Proof. (i) Suppose that

$$\lim_{n\to\infty}[\omega(n)-\nu(n)][\omega_0(n)-\nu_0(n)]^{-1} = \mu > 0.$$

For a given $\varepsilon > 0$, consider

$$\begin{aligned}&\{\nu_0(n) < k \le \omega_0(n) : |d(\xi_k, a) - d(a,a)| \ge \varepsilon\} \\ &\quad \supseteq \{\nu(n) < k \le \omega(n) : |d(\xi_k, a) - d(a,a)| \ge \varepsilon\}.\end{aligned}$$

Using this inclusion, we have

$$\begin{aligned}&\frac{1}{\omega_0(n)-\nu_0(n)}\left|\{\nu_0(n) < k \le \omega_0(n) : |d(\xi_k, a) - d(a,a)| \ge \varepsilon\}\right| \\ &\quad \ge [\omega(n)-\nu(n)][\omega_0(n)-\nu_0(n)]^{-1} \\ &\qquad \cdot \frac{1}{\omega(n)-\nu(n)}\left|\{\nu(n) < k \le \omega(n) : |d(\xi_k, a) - d(a,a)| \ge \varepsilon\}\right|.\end{aligned}$$

As $n \to \infty$, the left-hand side shows that $\xi_k \to a$ in the sense of $\mathrm{DS}_{d,\nu_0,\omega_0}$ implies $\xi_k \to a$ in the sense of $\mathrm{DS}_{d,\nu,\omega}$. Thus, $\mathrm{DS}_{d,\nu_0,\omega_0} \subseteq \mathrm{DS}_{d,\nu,\omega}$.

(ii) The proof is omitted since it relies on analogous reasoning, achieved by interchanging the roles of $\{\nu(n)\}, \{\omega(n)\}$ with $\{\nu_0(n)\}, \{\omega_0(n)\}$ under the specified condition

$$\lim_{n\to\infty}[\omega_0(n)-\nu_0(n)][\omega(n)-\nu(n)]^{-1} = 1.$$

∎

Theorem 3.10. *Let $\{\nu(n)\}$, $\{\omega(n)\}$, $\{\nu_0(n)\}$, and $\{\omega_0(n)\}$ be sequences of non-negative integers satisfying the conditions*

$$\nu_0(n) < \nu(n) < \omega(n) < \omega_0(n), \quad \forall n \in \mathbb{N}.$$

Then, the following statements hold:

(i) If $\lim_{n\to\infty}[\omega(n)-\nu(n)][\omega_0(n)-\nu_0(n)]^{-1} = \mu > 0$, and if a sequence is strongly Dwd_{d,ν_0,ω_0}-summable to ξ, then it is also $DS_{d,\nu,\omega}$-convergent to ξ.

(ii) If $\lim_{n\to\infty}[\omega_0(n) - \nu_0(n)][\omega(n) - \nu(n)]^{-1} = 1$, and if $\{\xi_k\}$ is a bounded sequence such that ξ_k is $DS_{d,\nu,\omega}$-convergent to ξ, then it follows that $\{\xi_k\}$ is strongly Dwd_{d,ν_0,ω_0}-summable to ξ.

Proof. (i) This follows directly from the definition and is omitted.

(ii) Suppose that $\mathrm{DS}_{d,\nu,\omega}$-$\lim \xi_k = \xi$ and that $\{\xi_k\}$ is a bounded sequence. Then, there exists some constant $M > 0$ such that $d(\xi_k, \xi) < M$ for all k. Given $\varepsilon > 0$, we can express

$$\frac{1}{\omega_0(n) - \nu_0(n)} \sum_{k=\nu_0(n)+1}^{\omega_0(n)} d(\xi_k, \xi) = \frac{1}{\omega_0(n) - \nu_0(n)} \sum_{k=\omega(n)+1}^{\omega_0(n)} d(\xi_k, \xi) + \frac{1}{\omega_0(n) - \nu_0(n)} \sum_{k=\nu(n)+1}^{\omega(n)} d(\xi_k, \xi).$$

Using the boundedness of $d(\xi_k, \xi)$, we obtain

$$\frac{1}{\omega_0(n) - \nu_0(n)} \sum_{k=\nu_0(n)+1}^{\omega_0(n)} d(\xi_k, \xi) \leq \left([\omega_0(n) - \omega(n)][\omega_0(n) - \nu_0(n)]^{-1}\right) M + \frac{1}{\omega_0(n) - \nu_0(n)} \sum_{k=\nu(n)+1}^{\omega(n)} d(\xi_k, \xi).$$

By the given assumption, the term

$$[\omega_0(n) - \nu_0(n)][\omega(n) - \nu(n)]^{-1} - 1$$

contributes to the first part, while the second term satisfies

$$\frac{1}{\omega(n) - \nu(n)} \sum_{k=\nu(n)+1}^{\omega(n)} d(\xi_k, \xi) < \varepsilon.$$

Thus, the sequence is strongly $\mathrm{Dwd}_{d,\nu_0,\omega_0}$-summable to ξ, completing the proof. ■

Definition 3.11. Let (S, d) be a pms. A sequence $\{\xi_k\}$ in S is called *strongly D-r-Cesàro summable* to $\xi \in S$ if the following condition holds,

$$\lim_{n\to\infty} \frac{1}{\omega(n) - \nu(n)} \sum_{k=\nu(n)+1}^{\omega(n)} |d(\xi_k, \xi) - d(\xi, \xi)|^r = 0.$$

In this case, we write

$$\mathrm{Dwd}^r_{d,\nu,\omega}\text{-} \lim \xi_k = \xi \quad \text{or} \quad \xi_k \to \xi \ (\mathrm{Dwd}^r_{d,\nu,\omega}).$$

Theorem 3.12. *Let (S,d) be a pms, and let $\{\xi_k\}$ be a sequence in S. If $\{\xi_k\}$ is strongly D-r-Cesàro summable to an element $\xi \in S$, then it is also statistically convergent to ξ. Conversely, if $\{\xi_k\}$ is bounded and statistically convergent to ξ, then it is strongly D-r-Cesàro summable to ξ.*

Proof. Suppose $\{\xi_k\}$ is strongly D-r-Cesàro summable to ξ. Then, for every $\varepsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{\omega(n)-\nu(n)} \sum_{k=\nu(n)+1}^{\omega(n)} |d(\xi_k,\xi) - d(\xi,\xi)|^r = 0.$$

Define $\Lambda_n = \{k : |d(\xi_k,\xi) - d(\xi,\xi)| \geq \varepsilon\}$. By the summability assumption,

$$\sum_{k\in\Lambda_n} |d(\xi_k,\xi) - d(\xi,\xi)|^r \geq |\Lambda_n| \cdot \varepsilon^r.$$

Dividing by $\omega(n)-\nu(n)$ and taking the limit, it follows that the density of Λ_n approaches zero, proving statistical convergence.

Conversely, assume $\{\xi_k\}$ is bounded and statistically convergent to ξ, meaning there exists $M > 0$ such that $|d(\xi_k,\xi) - d(\xi,\xi)| < M$ for all k. Define

$$\Lambda_n = \{k : |d(\xi_k,\xi) - d(\xi,\xi)| > (\varepsilon/2)^{1/r}\}.$$

Since statistical convergence implies $\lim_{n\to\infty} \frac{|\Lambda_n|}{\omega(n)-\nu(n)} = 0$, for sufficiently large n,

$$\frac{|\Lambda_n|}{\omega(n)-\nu(n)} < \frac{\varepsilon}{2M^r}.$$

Now, consider the summation

$$\frac{1}{\omega(n)-\nu(n)} \sum_{k=\nu(n)+1}^{\omega(n)} |d(\xi_k,\xi) - d(\xi,\xi)|^r.$$

Splitting this into contributions from Λ_n and its complement, the first term satisfies

$$\frac{|\Lambda_n|}{\omega(n)-\nu(n)} M^r < \frac{\varepsilon}{2}.$$

For $k \notin \Lambda_n$, we have $|d(\xi_k,\xi) - d(\xi,\xi)|^r \leq (\varepsilon/2)$, leading to

$$\frac{1}{\omega(n)-\nu(n)} \sum_{k\notin\Lambda_n} |d(\xi_k,\xi) - d(\xi,\xi)|^r < \frac{\varepsilon}{2}.$$

Thus, summing both parts gives

$$\frac{1}{\omega(n)-\nu(n)} \sum_{k=\nu(n)+1}^{\omega(n)} |d(\xi_k,\xi) - d(\xi,\xi)|^r < \varepsilon.$$

This proves strong D-r-Cesàro summability. ∎

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