ISSN 2985-7619

Hilbert Journal of Mathematical Analysis
Volume 3 Number 2 (2025)
Pages 012–018

https://hilbertjma.org

Intermediate Spaces on Weak Type Discrete Morrey Spaces

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Abstract In this article, we discuss inclusion between a discrete Morrey space and a weak discrete Morrey space, as well as inclusion between two weak discrete Morrey spaces. By the inclusion properties of weak discrete Morrey spaces, we have intermediate spaces for the trivial case. Using the inclusion relation of discrete Morrey spaces and weak discrete Morrey spaces, we obtain that for the nontrivial case, there is no weak discrete Morrey space between Banach pairs of weak discrete Morrey spaces except for the two weak discrete Morrey spaces itself.

MSC: 28C20; 43A15; 46B42; 54D45

Keywords: discrete Morrey spaces; weak discrete Morrey spaces; inclusion; intermediate spaces

Received: 28-02-2025 / Accepted: 11-04-2025 / Published: 05-05-2025
DOI: https://doi.org/10.62918/hjma.v3i2.33

1. Introduction

Morrey spaces are certain generalizations of Lebesgue spaces, and they were introduced by Morrey in [14]. In recent notation, these spaces are defined as follows. For $1 \leq p \leq q < \infty$, the Morrey space $\mathcal{M}^p_q = \mathcal{M}^p_q(\mathbb{R}^n)$ is defined to be the set of all L^p-locally integrable functions f such that

$$\sup_{a\in\mathbb{R}^n, r>0} |B(a,r)|^{\frac{1}{q}-\frac{1}{p}} \left(\int_{B(a,r)} |f(x)|^p \, dx\right)^{1/p}$$

is finite. Here, $B(a,r)$ denotes the ball of radius r and is centered at a. The notation $|B(a,r)|$ is the Lebesgue measure of $B(a,r)$. Observe that, for $p = q$, we may recover Lebesgue spaces as special cases of Morrey spaces. However, we have a proper inclusion $L^q \subsetneq \mathcal{M}^p_q$ if $p \neq q$. Although Morrey spaces were introduced a long time ago, there is still much recent research about these spaces (see [1, 18, 19] and references therein). Among these research, there are two main properties in many studies about Morrey spaces, namely, inclusion property and interpolation property of Morrey spaces. The

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results about the inclusion property of Morrey spaces can be found in [2, 4, 5, 7, 9, 15–17, 20]). The interpolation property of Morrey spaces can be seen in [11–13].

In this paper, we use inclusion results of weak-type discrete Morrey spaces to investigate intermediate spaces of these spaces. Let us first recall the definition of discrete Morrey spaces.

Definition 1.1. Let $1 \leq p \leq q < \infty$. Discrete Morrey space $\ell_q^p = \ell_q^p(\mathbb{Z})$ is the set of all sequences $x = (x_j)_{j\in\mathbb{Z}}$ such that

$$\sup_{m\in\mathbb{Z}, N\in\omega} |S_{m,N}|^{\frac{1}{q}-\frac{1}{p}} \left(\sum_{j\in S_{m,N}} |x_j|^p \right)^{\frac{1}{p}} < \infty$$

where $\omega = \mathbb{N} \cup \{0\}$ and $S_{m,N} = \{m-N, \ldots, m, \ldots, m+N\}$. This space is a Banach space with the norm

$$\|x\|_{\ell_q^p} := \sup_{m\in\mathbb{Z}, N\in\omega} |S_{m,N}|^{\frac{1}{q}-\frac{1}{p}} \left(\sum_{j\in S_{m,N}} |x_j|^p \right)^{\frac{1}{p}}.$$

The weak type of discrete Morrey space is defined as follow.

Definition 1.2. Let $1 \leq p \leq q < \infty$. Weak type discrete Morrey space $w\ell_q^p = w\ell_q^p(\mathbb{Z})$ is the set of all sequences $x = (x_j)_{j\in\mathbb{Z}}$ for which the quasi-norm

$$\|x\|_{w\ell_q^p} := \sup_{m\in\mathbb{Z}, N\in\omega, \gamma>0} \gamma |S_{m,N}|^{\frac{1}{q}-\frac{1}{p}} |\{j \in S_{m,N} : |x_j| > \gamma\}|^{\frac{1}{p}}$$

is finite.

Weak type discrete Morrey space is complete. Furthermore, in [3] it is shown that $\ell_q^p \subseteq w\ell_q^p$ with $\|x\|_{w\ell_q^p} \leq \|x\|_{\ell_q^p}$ for all $x \in \ell_q^p$.

Inclusion relation on discrete Morrey spaces and its weak type has been studied in [2, 3, 6, 8] Moreover, some relations between discrete Morrey space and its weak type have been obtained in [3]. These results are as follows.

Theorem 1.3. *Let $1 \leq p_1 \leq q_1 < \infty$ and $1 \leq p_2 \leq q_2 < \infty$. Then $\ell_{q_1}^{p_1} \subseteq \ell_{q_2}^{p_2}$ and $w\ell_{q_1}^{p_1} \subseteq w\ell_{q_2}^{p_2}$ if and only if $q_1 \leq q_2$ and $\frac{p_2}{q_2} \leq \frac{p_1}{q_1}$.*

Theorem 1.4. *Let $1 \leq p_1 \leq q_1 < \infty$ and $1 \leq p_2 \leq q_2 < \infty$. The inclusion $w\ell_{q_1}^{p_1} \subseteq \ell_{q_2}^{p_2}$ implies $q_1 \leq q_2$ and $\frac{p_2}{q_2} \leq \frac{p_1}{q_1}$.*

Theorem 1.5. *Let $1 \leq p_1 \leq q_1 < \infty$ and $1 \leq p_2 \leq q_2 < \infty$. If $q_1 \leq q_2$ and $p_2 < p_1$, then $w\ell_{q_1}^{p_1} \subseteq \ell_{q_2}^{p_2}$.*

Theorem 1.6. *Let $1 \leq p_1 \leq q < \infty$ and $1 \leq p_2 \leq q < \infty$. If $w\ell_q^{p_1} \subseteq \ell_q^{p_2}$, then $p_2 < p_1$.*

One of the things studied in Banach spaces is their intermediate space. According to [10], given Banach space X_1 and X_2, Banach spaces X that satisfy

$$X_1 \cap X_2 \subseteq X \subseteq X_1 + X_2$$

is called intermediate spaces between X_1 and X_2. Since weak type discrete Morrey space is not Banach space, we generalized this definition of intermediate space for quasi-Banach space.

Intermediate spaces for discrete Morrey spaces have been studied by Yudatama [21]. It is found that if two discrete Morrey spaces $\ell^{p_1}_{q_1}$ and $\ell^{p_2}_{q_2}$ do not have inclusion relation, the intermediate spaces for Banach couple $\ell^{p_1}_{q_1}$ and $\ell^{p_2}_{q_2}$ that are discrete Morrey spaces are only those spaces itself. The reason why there are no other intermediate spaces can be seen in the theorems below.

Theorem 1.7. *Let $1 \le p \le q < \infty$, $1 \le p_1 < q_1 < \infty$, $1 \le p_2 \le q_2 < \infty$, $q_1 < q_2$, and $\frac{p_1}{q_1} < \frac{p_2}{q_2}$. If $q_1 < q$ and $\frac{p}{q} < \frac{p_2}{q_2}$, then $\ell^p_q \not\subseteq \ell^{p_1}_{q_1} + \ell^{p_2}_{q_2}$.*

Theorem 1.8. *Let $1 \le p \le q < \infty$, $1 \le p_1 < q_1 < \infty$, $1 \le p_2 \le q_2 < \infty$, $q_1 < q_2$, and $\frac{p_1}{q_1} < \frac{p_2}{q_2}$. If $q < q_1$ or $\frac{p_2}{q_2} < \frac{p}{q}$, then $\ell^{p_1}_{q_1} \cap \ell^{p_2}_{q_2} \not\subseteq \ell^p_q$.*

Theorem 1.9. *Let $1 \le p \le q < \infty$, $1 \le p_1 < q_1 < \infty$, $1 \le p_2 \le q_2 < \infty$, $q_1 < q_2$, and $\frac{p_1}{q_1} < \frac{p_2}{q_2}$. If $q_1 = q$ and $\frac{p_1}{q_1} < \frac{p}{q} < \frac{p_2}{q_2}$, then $\ell^{p_1}_{q_1} \cap \ell^{p_2}_{q_2} \not\subseteq \ell^p_q$.*

Theorem 1.10. *Let $1 \le p \le q < \infty$, $1 \le p_1 < q_1 < \infty$, $1 \le p_2 \le q_2 < \infty$, $q_1 < q_2$, and $\frac{p_1}{q_1} < \frac{p_2}{q_2}$. If $q_1 < q < q_2$ and $\frac{p}{q} = \frac{p_2}{q_2}$, then $\ell^{p_1}_{q_1} \cap \ell^{p_2}_{q_2} \not\subseteq \ell^p_q$.*

The aim of this paper is to extend the result about intermediate spaces on discrete Morrey spaces to their weak counterparts. In order to do that, we first prove that $q_1 < q_2$ implies $w\ell^p_{q_1} \subseteq \ell^p_{q_2}$. Using this relation, results that have been obtained in discrete Morrey space will be adapted for its weak type.

2. Main Results

Theorem 2.1. *Let $1 \le p \le q_1 < \infty$ and $1 \le p \le q_2 < \infty$. If $q_1 < q_2$, then $w\ell^p_{q_1} \subseteq \ell^p_{q_2}$.*

Proof. Let $q_3 = q_2$ and $p_3 = \frac{q_3}{q_1}p$. We get $q_1 < q_3$, $q_3 = q_2$, $p < p_3$, and $\frac{p}{q_1} = \frac{p_3}{q_3}$ which means $w\ell^p_{q_1} \subseteq w\ell^{p_3}_{q_3}$ and $w\ell^{p_3}_{q_3} \subseteq \ell^p_{q_2}$. Therefore $w\ell^p_{q_1} \subseteq \ell^p_{q_2}$. ∎

Now, we will show that there are no intermediate spaces between two weak type discrete Morrey spaces that have no inclusion relation, in the form of weak type discrete Morrey space, except for those spaces itself. To show that there is no intermediate space between $w\ell^{p_1}_{q_1}$ and $w\ell^{p_2}_{q_2}$, we will check for every possible value $1 \le p \le q < \infty$.

Theorem 2.2. *Let $1 \le p \le q < \infty$, $1 \le p_1 < q_1 < \infty$, $1 \le p_2 \le q_2 < \infty$, $q_1 < q_2$, and $\frac{p_1}{q_1} < \frac{p_2}{q_2}$. If $q_1 < q$ and $\frac{p}{q} < \frac{p_2}{q_2}$, then $w\ell^p_q \not\subseteq w\ell^{p_1}_{q_1} + w\ell^{p_2}_{q_2}$.*

Proof. Let $q_3 = \frac{q_1+q}{2}$, $p_3 = p_1$, and $p_4 = p_2$. Let $\varepsilon > 0$ such that $\frac{p}{q} < \frac{p_2}{q_2+\varepsilon}$ and let $q_4 = q_2 + \varepsilon$. Then $q_3 < q$ and $\frac{p}{q} < \frac{p_4}{q_4}$. Using Theorems 1.7 and 2.1 we get that $w\ell^{p_1}_{q_1} \subseteq \ell^{p_3}_{q_3}$, $w\ell^{p_2}_{q_2} \subseteq \ell^{p_4}_{q_4}$, and $\ell^p_q \not\subseteq \ell^{p_3}_{q_3} + \ell^{p_4}_{q_4}$. Since $\ell^p_q \subseteq w\ell^p_q$ and $w\ell^{p_1}_{q_1} + w\ell^{p_2}_{q_2} \subseteq \ell^{p_3}_{q_3} + \ell^{p_4}_{q_4}$, it is clear that $w\ell^p_q \not\subseteq w\ell^{p_1}_{q_1} + w\ell^{p_2}_{q_2}$. ∎

Theorem 2.3. *Let $1 \le p \le q < \infty$, $1 \le p_1 < q_1 < \infty$, $1 \le p_2 \le q_2 < \infty$, $q_1 < q_2$, and $\frac{p_1}{q_1} < \frac{p_2}{q_2}$. If $q < q_1$, then $w\ell^{p_1}_{q_1} \cap w\ell^{p_2}_{q_2} \not\subseteq w\ell^p_q$.*

Proof. Let $\varepsilon > 0$ such that $q+\varepsilon < q_1$. Using Theorems 1.8 and 2.1 we get that $w\ell^p_q \subseteq \ell^p_{q+\varepsilon}$ and $\ell^{p_1}_{q_1} \cap \ell^{p_2}_{q_2} \not\subseteq \ell^p_{q+\varepsilon}$. Since $\ell^{p_1}_{q_1} \cap \ell^{p_2}_{q_2} \subseteq w\ell^{p_1}_{q_1} \cap w\ell^{p_2}_{q_2}$, it is clear that $w\ell^{p_1}_{q_1} \cap w\ell^{p_2}_{q_2} \not\subseteq w\ell^p_q$. ∎

Theorem 2.4. *Let $1 \le p \le q < \infty$, $1 \le p_1 < q_1 < \infty$, $1 \le p_2 \le q_2 < \infty$, $q_1 < q_2$, and $\frac{p_1}{q_1} < \frac{p_2}{q_2}$. If $\frac{p_2}{q_2} < \frac{p}{q}$, then $w\ell^{p_1}_{q_1} \cap w\ell^{p_2}_{q_2} \not\subseteq w\ell^p_q$.*

Proof. Let $\varepsilon > 0$ such that $\frac{p_2}{q_2} < \frac{p}{q+\varepsilon}$. Using Theorems 1.8 and 2.1 we get that $w\ell^p_q \subseteq \ell^p_{q+\varepsilon}$ and $\ell^{p_1}_{q_1} \cap \ell^{p_2}_{q_2} \not\subseteq \ell^p_{q+\varepsilon}$. Since $\ell^{p_1}_{q_1} \cap \ell^{p_2}_{q_2} \subseteq w\ell^{p_1}_{q_1} \cap w\ell^{p_2}_{q_2}$, it is clear that $w\ell^{p_1}_{q_1} \cap w\ell^{p_2}_{q_2} \not\subseteq w\ell^p_q$. ∎

Theorem 2.5. *Let $1 \le p \le q < \infty$, $1 \le p_1 < q_1 < \infty$, $1 \le p_2 \le q_2 < \infty$, $q_1 < q_2$, and $\frac{p_1}{q_1} < \frac{p_2}{q_2}$. If $q_1 = q$ and $\frac{p_1}{q_1} < \frac{p}{q} < \frac{p_2}{q_2}$, then $w\ell^{p_1}_{q_1} \cap w\ell^{p_2}_{q_2} \not\subseteq w\ell^p_q$.*

Proof. Let $\varepsilon > 0$ such that $\frac{p_1}{q_1} < \frac{p-\varepsilon}{q}$. Using Theorems 1.5 and 1.9 we get that $w\ell^p_q \subseteq \ell^{p-\varepsilon}_q$ and $\ell^{p_1}_{q_1} \cap \ell^{p_2}_{q_2} \not\subseteq \ell^{p-\varepsilon}_q$. Since $\ell^{p_1}_{q_1} \cap \ell^{p_2}_{q_2} \subseteq w\ell^{p_1}_{q_1} \cap w\ell^{p_2}_{q_2}$, it is clear that $w\ell^{p_1}_{q_1} \cap w\ell^{p_2}_{q_2} \not\subseteq w\ell^p_q$. ■

For the last case, that is when $q_1 < q < q_2$ and $\frac{p}{q} = \frac{p_2}{q_2}$, we will prove that $w\ell^{p_1}_{q_1} \cap w\ell^{p_2}_{q_2} \not\subseteq w\ell^p_q$ by constructing a sequence that is in $w\ell^{p_1}_{q_1} \cap w\ell^{p_2}_{q_2}$ but not in $w\ell^p_q$. This sequence is introduced in [21] to prove $\ell^{p_1}_{q_1} \cap \ell^{p_2}_{q_2} \not\subseteq \ell^p_q$ when $q_1 < q < q_2$ and $\frac{p}{q} = \frac{p_2}{q_2}$.

Definition 2.6. Let $u \in \mathbb{R}$, $v, w \in \mathbb{N}$, $u \ge 1$, and $v > 2$. Let

$$H_1 = \bigcup_{k\in\mathbb{N},k\ge 2} \{2^{k(v+w)} - i2^{kw} : i = 0, 1, \dots, 2^{k(v-2)}\},$$
$$H_2 = \{-2^{v+w}, \dots, 0 \dots, 2^{v+w}\},$$
$$H_3 = \bigcup_{k\in\mathbb{N},k\ge 2} \{-2^{k(v+w)} + i2^{kw} : i = 0, 1, \dots, 2^{k(v-2)}\},$$

and

$$H = H_1 \cup H_2 \cup H_3.$$

Let $\varphi : \mathbb{Z} \to H$ be bijective increasing odd function, that is $\varphi(j) = -\varphi(-j)$ and $\varphi(|j|) < \varphi(|j| + 1)$. Define a sequence $x^{(u,v,w)} = (x^{(u,v,w)}_j)_{j\in\mathbb{Z}}$ as

$$x^{(u,v,w)}_j = \begin{cases} 1, & j = 0, \\ |\varphi^{-1}(j)|^{-\frac{1}{u}}, & j \in H\backslash\{0\} \\ 0, & \text{others.} \end{cases}$$

Yudatama [21] has shown that $x^{(u,v,w)}$ is an element of ℓ^p_q if and only if

$$(v-2)\left(\frac{1}{q} - \frac{1}{u}\right) - w\left(\frac{1}{p} - \frac{1}{q}\right) \le 0.$$

We will show that this condition applies for $w\ell^p_q$ as well.

Lemma 2.7. *Let $1 \le p \le q < \infty$, $u \in \mathbb{R}$, $v, w \in \mathbb{N}$, $u \ge 1$, and $v > 2$. The sequence $x^{(u,v,w)}$ is in $w\ell^p_q$ if and only if u, v, and w satisfy*

$$(v-2)\left(\frac{1}{q} - \frac{1}{u}\right) - w\left(\frac{1}{p} - \frac{1}{q}\right) \le 0. \tag{2.1}$$

Proof. Since $\ell^p_q \subseteq w\ell^p_q$, it is clear that if u, v, and w satisfy (2.1) then $x^{(u,v,w)} \in \ell^p_q \subseteq w\ell^p_q$. For the reader's convenience, we will show the sketch of the proof. Let $m \in \mathbb{Z}$ and $N \in \omega$. The sequence $x^{(u,v,w)}$ is symmetric. Therefore, we only consider $m \ge 0$. For $N \le 2^{v+w}$ we have

$$|S_{m,N}|^{\frac{1}{q}-\frac{1}{p}} \left(\sum_{j\in S_{m,N}} |x^{(u,v,w)}_j|^p\right)^{\frac{1}{p}} \le (2N+1)^{\frac{1}{q}-\frac{1}{p}} (2N+1)^{\frac{1}{p}} \le \left(1 + 2^{1+v+w}\right)^{\frac{1}{q}}.$$

For $m = 0$ and $N > 2^{v+w}$ we have

$$|S_{0,N}|^{\frac{1}{q}-\frac{1}{p}} \left(\sum_{j\in S_{0,N}} |x^{(u,v,w)}_j|^p\right)^{\frac{1}{p}} \le C2^{r(v+w)\left(\frac{1}{q}-\frac{1}{p}\right)} 2^{r(v-2)\left(\frac{u-p}{up}\right)}$$

for some C. Since u, v, and w satisfy (2.1), the right-hand side is bounded by C. For $m \leq 3N$ and $N > 2^{v+w}$, we have

$$\sum_{j\in S_{m,N}} |x_j^{(u,v,w)}|^p \leq \left(1 + 2^{3(v-2)}\right)\left(\sum_{j\in S_{0,N}} |x_j^{(u,v,w)}|^p\right).$$

Therefore the boundedness of $|S_{m,N}|^{\frac{1}{q}-\frac{1}{p}}\left(\sum_{j\in S_{m,N}} |x_j^{(u,v,w)}|^p\right)^{\frac{1}{p}}$ follows from the boundedness of $|S_{0,N}|^{\frac{1}{q}-\frac{1}{p}}\left(\sum_{j\in S_{0,N}} |x_j^{(u,v,w)}|^p\right)^{\frac{1}{p}}$. Lastly, for $m > 3N$ and $N > 2^{v+w}$ we have

$$|S_{m,N}|^{\frac{1}{q}-\frac{1}{p}}\left(\sum_{j\in S_{m,N}} |x_j^{(u,v,w)}|^p\right)^{\frac{1}{p}} \leq 2^{\frac{1}{p}+\frac{v-2}{q}} 2^{r\left((v-2)\left(\frac{1}{q}-\frac{1}{u}\right)-w\left(\frac{1}{p}-\frac{1}{q}\right)\right)}$$

and it is bounded since u, v, and w satisfy (2.1).

Now we show that if u, v, and w not satisfy (2.1) then $x^{(u,v,w)} \notin w\ell^p_q$. Consider discrete intervals in the form of $[2^{r(v+w)} - 2^{r(v+w-2)}, 2^{r(v+w)}]$ with $r \in \mathbb{N}$ and $r > 2$. These intervals contain exactly $2^{r(v-2)}+1$ nonzero terms with the smallest nonzero terms is greater than $c2^{-\frac{(r+1)(v-2)}{u}}$ with $c = 2^{-\frac{v+w+2}{u}}$. For every $r \in \mathbb{N}$ with $r \geq 2$, let

$$S_{m,N} = [2^{r(v+w)} - 2^{r(v+w-2)}, 2^{r(v+w)}]$$

and $\gamma = c2^{-\frac{(r+1)(v-2)}{u}}$. We get

$$\begin{aligned}\gamma|S_{m,N}|^{\frac{1}{q}-\frac{1}{p}}|\{j \in S_{m,N} : |x_j| > \gamma\}|^{\frac{1}{p}} &= \gamma|2^{r(v+w-2)} + 1|^{\frac{1}{q}-\frac{1}{p}}|2^{r(v-2)} + 1|^{\frac{1}{p}} \\ &\geq \gamma|2^{r(v+w-2)+1}|^{\frac{1}{q}-\frac{1}{p}}|2^{r(v-2)}|^{\frac{1}{p}} \\ &\geq c2^{\frac{1}{q}-\frac{1}{p}-\frac{v-2}{u}} 2^{r((v-2)(\frac{1}{q}-\frac{1}{u})+w(\frac{1}{q}-\frac{1}{p}))}.\end{aligned}$$

Since u, v, and w do not satisfy (2.1), this expressions is not bounded which means $x^{(u,v,w)} \notin w\ell^p_q$. ∎

From this lemma we can conclude that $x^{(u,v,w)} \in \ell^p_q$ if and only if $x^{(u,v,w)} \in w\ell^p_q$. With this fact, we get the following result.

Theorem 2.8. *Let $1 \leq p \leq q < \infty$, $1 \leq p_1 < q_1 < \infty$, $1 \leq p_2 \leq q_2 < \infty$, $q_1 < q_2$, and $\frac{p_1}{q_1} < \frac{p_2}{q_2}$. If $q_1 < q < q_2$ and $\frac{p}{q} = \frac{p_2}{q_2}$, then $w\ell^{p_1}_{q_1} \cap w\ell^{p_2}_{q_2} \not\subseteq w\ell^p_q$.*

Proof. From [21] we have u, v, and w such that $x^{(u,v,w)} \in \ell^{p_1}_{q_1} \cap \ell^{p_2}_{q_2}$ but $x^{(u,v,w)} \notin \ell^p_q$. With the same u, v, and w we get that $x^{(u,v,w)} \in w\ell^{p_1}_{q_1} \cap w\ell^{p_2}_{q_2}$ but $x^{(u,v,w)} \notin w\ell^p_q$. Therefore $w\ell^{p_1}_{q_1} \cap w\ell^{p_2}_{q_2} \not\subseteq w\ell^p_q$. ∎

3. Concluding Remarks

Based on main results, we obtain that given two weak type discrete Morrey spaces $w\ell^{p_1}_{q_1}$ and $w\ell^{p_2}_{q_2}$ that has no inclusion relation, there is no intermediate space in the form of weak type discrete Morrey space except for $w\ell^{p_1}_{q_1}$ and $w\ell^{p_2}_{q_2}$.

Acknowledgements

This research is supported by FMIPA-ITB under FMIPA-PPMI-KK award number 616AO/IT1.CO2/KU/2024.

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