ISSN 2985-7619

Hilbert Journal of Mathematical Analysis
Volume 3 Number 2 (2025)
Pages 028–033

https://hilbertjma.org

Inclusion between Bourgain-Morrey spaces and their weak types

Aria Pratama[1,*], **Denny Ivanal Hakim**[2]

[1] *Faculty of Mathematics and Natural Sciences, Bandung Institute of Technology, Bandung 40132, Indonesia*
e-mail: ariapratama@sci.ui.ac.id

[2] *Faculty of Mathematics and Natural Sciences, Bandung Institute of Technology, Bandung 40132, Indonesia*
e-mail: dhakim@itb.ac.id

Abstract This paper aims to explore the embedding relation between Bourgain-Morrey spaces and weak Bourgain-Morrey spaces. Bourgain-Morrey spaces are important function spaces in harmonic analysis. In this article, we introduce weak Bourgain-Morrey spaces as a certain generalization of weak Morrey spaces. We investigate the necessary and sufficient conditions for functions in weak Bourgain-Morrey spaces to also be elements in Bourgain-Morrey spaces, and vice versa. Our results are the inclusion of weak Bourgain-Morrey spaces and the inclusion of weak Bourgain-Morrey spaces into a certain Bourgain-Morrey space. These results generalize the inclusion result in Morrey spaces, weak Morrey spaces, and Bourgain-Morrey spaces.

MSC: 42B35; 46E30

Keywords: Bourgain-Morrey space; Morrey space; weak Morrey space; inclusion property

Received: 28-02-2025 / Accepted: 14-07-2025 / Published: 23-07-2025
DOI: https://doi.org/10.62918/hjma.v3i2.35

1. Introduction

Morrey spaces were first introduced by C. B. Morrey in [5]. Let $1 \le q \le p < \infty$. Define Morrey space $\mathcal{M}^p_q = \mathcal{M}^p_q(\mathbb{R}^n)$ as a set of all $f \in L^q_{loc}(\mathbb{R}^n)$ where

$$\|f\|_{\mathcal{M}^p_q} = \sup_{B(a,r)} |B(a,r)|^{\frac{1}{p}-\frac{1}{q}} \left(\int_{B(a,r)} |f(x)|^q \, dx \right)^{\frac{1}{q}} < \infty.$$

Morrey spaces are a generalization of Lebesgue spaces; hence $\mathcal{M}^p_p$ is the Lebesgue space L^p. Inclusion in Morrey spaces has been discussed in [1–3, 8]. Inclusion of the generalization of Morrey spaces has also been widely discussed. In [8], inclusion in mixed Morrey spaces has been introduced and discussed, and in [2], inclusion in weak Morrey spaces has been introduced and discussed. In [4], the Bourgain-Morrey spaces are introduced, which are

*Corresponding author.

a generalization of the Morrey spaces. Adapting Definition 2.1, we define weak Bourgain-Morrey spaces in Definition 3.1 as a generalization of the Bourgain-Morrey spaces. Our main result in this paper is the inclusion of weak Bourgain-Morrey spaces.

2. Preliminaries

Adapting the extension of the Lebesgue spaces L^p into a weak Lebesgue space wL^p, weak Morrey spaces can also be defined. Weak Morrey spaces were introduced by Gunawan et al. in [2] with the following definition.

Definition 2.1. [2] For $1 \le q \le p < \infty$, *weak Morrey space* $w\mathcal{M}_q^p = w\mathcal{M}_q^p(\mathbb{R}^n)$ is the set of all measurable function f for which $\|f\|_{w\mathcal{M}_q^p} < \infty$ where

$$||f||_{w\mathcal{M}_q^p} = \sup_{\gamma>0} \left\| \gamma \chi_{\{|f|>\gamma\}} \right\|_{\mathcal{M}_q^p}.$$

Based on definition of Morrey space in [7], $||f||_{w\mathcal{M}_q^p}$ can be expressed as

$$||f||_{w\mathcal{M}_q^p} = \sup_{a\in\mathbb{R}^n, r>0, \gamma>0} |B(a,r)|^{\frac{1}{p}-\frac{1}{q}} \left(\int_{B(a,r)} \left|\gamma\chi_{\{|f|>\gamma\}}(x)\right|^q dx \right)^{\frac{1}{q}}.$$

In this definition, $B(a,r)$ is an open ball centered at $a \in \mathbb{R}^n$ with radius $r > 0$ and $|B(a,r)|$ denotes its Lebesgue measure. By using the inequality $\left|\gamma\chi_{\{|f|>\gamma\}}\right| \le |f|$ for every $\gamma > 0$, we have $\mathcal{M}_q^p \subseteq w\mathcal{M}_q^p$. The following inclusion properties are discussed in [2].

Theorem 2.2. [2] *For* $1 \le q_1 \le q_2 \le p < \infty$, *the following inclusion holds*

$$w\mathcal{M}_{q_2}^p \subseteq w\mathcal{M}_{q_1}^p.$$

If $q_1 < q_2$, *then*

$$w\mathcal{M}_{q_2}^p \subseteq \mathcal{M}_{q_1}^p.$$

Remark 2.3. Based on Theorem 2.2, it is concluded that for $1 \le q_1 < q_2 \le p < \infty$ holds

$$\mathcal{M}_{q_2}^p \subseteq w\mathcal{M}_{q_2}^p \subseteq \mathcal{M}_{q_1}^p.$$

.

Recall that Q_{vm} are dyadic cubes with a volume 2^{-vn} where

$$Q_{vm} = \prod_{j=1}^{n} \left[\frac{m_j}{2^v}, \frac{m_j+1}{2^v}\right) \text{ with } v \in \mathbb{Z}, m = (m_1, m_2 \dots, m_n) \in \mathbb{Z}^n.$$

Here, $\left[\frac{m_j}{2^v}, \frac{m_j+1}{2^v}\right) = \left\{x : \frac{m_j}{2^v} \le x < \frac{m_j+1}{2^v}\right\}$. Note that for $1 < q < p < \infty$ and $f \in L^q_{loc}$, the norm of the Morrey space $\mathcal{M}_q^p$ can be expressed as

$$\|f\|_{\mathcal{M}_q^p} \equiv \sup_{\substack{v\in\mathbb{Z}\\ m\in\mathbb{Z}^n}} |Q_{vm}|^{\frac{1}{p}-\frac{1}{q}} \left(\int_{Q_{vm}} |f(x)|^q \, dx \right)^{\frac{1}{q}}.$$

Bourgain-Morrey spaces were introduced in [4] as a generalization of Morrey spaces. There are several characteristics that are possessed by Bourgain-Morrey spaces but not by Morrey spaces. For example, Bourgain-Morrey spaces have the density property of the set of smooth functions with compact support, but this property does not hold in

Morrey spaces. We refer the reader to ([6]; Chapter 8). Let us recall the definition of Bourgain-Morrey spaces below.

Definition 2.4. [4] Let $1 \le q \le p < \infty$ and $1 \le r \le \infty$. Define $\mathcal{M}^p_{q,r} = \mathcal{M}^p_{q,r}(\mathbb{R}^n)$ as a set of all $f \in L^q_{loc}$ for which

$$\|f\|_{\mathcal{M}^p_{q,r}} = \left\| \left\{ |Q_{vm}|^{\frac{1}{p}-\frac{1}{q}} \left(\int_{B(a,r)} |f(x)|^q \, dx \right)^{\frac{1}{q}} \right\}_{v\in\mathbb{Z}, m\in\mathbb{Z}^n} \right\|_{\ell^r}$$

is finite. Here, $\|\{x_{\nu m}\}_{v\in\mathbb{Z},m\in\mathbb{Z}^n}\|_{\ell^r} = \begin{cases} \left(\sum\limits_{v\in\mathbb{Z},m\in\mathbb{Z}^n} |x_{\nu m}|^r \right)^{1/r}, & \text{if } 1 \le r < \infty, \\ \sup\limits_{v\in\mathbb{Z},m\in\mathbb{Z}^n} |x_{\nu m}|, & \text{if } r = \infty. \end{cases}$

Remark 2.5. For $r = \infty$, the Bourgain-Morrey space $\mathcal{M}^p_{q,r}$ coincides with the Morrey spaces, $\mathcal{M}^p_q = \mathcal{M}^p_{q,\infty}$. For $r < \infty$, $\mathcal{M}^p_{q,r}$ spaces are smaller than Morrey spaces $\mathcal{M}^p_q$. In other words, $\mathcal{M}^p_{q,r} \subseteq \mathcal{M}^p_q = \mathcal{M}^p_{q,\infty}$.

The following inclusion properties are discussed in [4] as follows.

Lemma 2.6. [4] *Let $1 \le q \le p < r_1 \le r_2 \le \infty$. Then $\mathcal{M}^p_{q,r_1} \subseteq \mathcal{M}^p_{q,r_2}$.*

Lemma 2.7. [4] *Let $1 \le q_2 \le q_1 \le p < r \le \infty$. Then $\mathcal{M}^p_{q_1,r} \subseteq \mathcal{M}^p_{q_2,r}$.*

3. Main Result

Adapting Definition 2.1, we define the weak Bourgain-Morrey spaces as follows.

Definition 3.1. Let $1 \le q \le p < \infty$ and $1 \le r \le \infty$. The *weak Bourgain-Morrey space* $w\mathcal{M}^p_q = w\mathcal{M}^p_q(\mathbb{R}^n)$ is the set of all measurable function f for which $\|f\|_{w\mathcal{M}^p_{q,r}} < \infty$ where

$$||f||_{w\mathcal{M}^p_{q,r}} := \sup_{\gamma>0} \left\|\gamma\chi_{\{|f|>\gamma\}}\right\|_{\mathcal{M}^p_{q,r}}.$$

By Definition 2.4, norm on Definition 3.1 can be stated as

$$\|f\|_{w\mathcal{M}^p_{q,r}} = \sup_{\gamma>0} \left\| \left\{ |Q_{vm}|^{\frac{1}{p}-\frac{1}{q}} \left(\int_{Q_{vm}} \left|\gamma\chi_{\{|f|>\gamma\}}\right|^q dx \right)^{\frac{1}{q}} \right\}_{v\in\mathbb{Z},m\in\mathbb{Z}^n} \right\|_{\ell^r}.$$

Similar to weak Morrey spaces, the following inclusions are obtained.

Lemma 3.2. *Let $1 \le q \le p < \infty$. Then $\mathcal{M}^p_{q,r} \subseteq w\mathcal{M}^p_{q,r}$ for all $f \in \mathcal{M}^p_{q,r}$.*

Proof. Let $f \in \mathcal{M}^p_q$. Because for every $\gamma > 0$ applies $\left|\gamma\chi_{\{|f|>\gamma\}}\right| \le |f|$, then we get

$$|Q_{vm}|^{\frac{1}{p}-\frac{1}{q}} \left(\int_{Q_{vm}} \left|\gamma\chi_{\{|f|>\gamma\}}(x)\right|^q dx \right)^{\frac{1}{q}} \le |Q_{vm}|^{\frac{1}{p}-\frac{1}{q}} \left(\int_{Q_{vm}} |f(x)|^q \, dx \right)^{\frac{1}{q}}.$$

As a result

$$\begin{aligned}
&\|f\|_{w\mathcal{M}^p_{q,r}} \\
&= \left\| \left\{ |Q_{vm}|^{\frac{1}{p}-\frac{1}{q}} \left(\int_{Q_{vm}} \left|\gamma\chi_{\{|f|>\gamma\}}(x)\right|^q dx \right)^{\frac{1}{q}} \right\}_{\substack{v\in\mathbb{Z}\\ m\in\mathbb{Z}^n}} \right\|_{l^r} \\
&\leq \left\| \left\{ |Q_{vm}|^{\frac{1}{p}-\frac{1}{q}} \left(\int_{Q_{vm}} |f(x)|^q dx \right)^{\frac{1}{q}} \right\}_{\substack{v\in\mathbb{Z}\\ m\in\mathbb{Z}^n}} \right\|_{l^r} \\
&= \|f\|_{\mathcal{M}^p_{q,r}} < \infty.
\end{aligned}$$

Therefore $f \in w\mathcal{M}^p_{q,r}$. ∎

Based on Lemma 2.6 and Lemma 2.7, we obtain several inclusion results between weak Bourgain-Morrey space as follows.

Lemma 3.3. *Let $1 \leq q \leq p < r_1 \leq r_2 \leq \infty$. Then $w\mathcal{M}^p_{q,r_1} \subseteq w\mathcal{M}^p_{q,r_2}$.*

Proof. Let $f \in w\mathcal{M}^p_{q,r_1}$. Based on Lemma 2.6 , we get $\|f\|_{\mathcal{M}^p_{q,r_2}} \leq C\|f\|_{\mathcal{M}^p_{q,r_1}}$ so that

$$\begin{aligned}
\left\|\gamma\chi_{\{|f|>\gamma\}}\right\|_{\mathcal{M}^p_{q,r_2}} &\leq C\left\|\gamma\chi_{\{|f|>\gamma\}}\right\|_{\mathcal{M}^p_{q,r_1}} \\
&\leq C\sup_{\gamma>0}\left\|\gamma\chi_{\{|f|>\gamma\}}\right\|_{\mathcal{M}^p_{q,r_1}} \\
&= C\|f\|_{w\mathcal{M}^p_{q,r_1}} \\
&< \infty.
\end{aligned}$$

Since the above inequality holds for every $\gamma > 0$, we have

$$\sup_{\gamma>0}\left\|\gamma\chi_{\{|f|>\gamma\}}\right\|_{\mathcal{M}^p_{q,r_2}} = ||f||_{w\mathcal{M}^p_{q,r_2}} < \infty.$$

Therefore $w\mathcal{M}^p_{q,r_1} \subseteq w\mathcal{M}^p_{q,r_2}$. ∎

Lemma 3.4. *Let $1 \leq q_2 \leq q_1 \leq p < r \leq \infty$. Then $w\mathcal{M}^p_{q_1,r} \subseteq w\mathcal{M}^p_{q_2,r}$.*

Proof. Let $f \in w\mathcal{M}^p_{q_1,r}$. Based on Lemma 2.7, we get $\|f\|_{\mathcal{M}^p_{q_2,r}} \leq \|f\|_{\mathcal{M}^p_{q_1,r}}$ so that

$$\begin{aligned}
\left\|\gamma\chi_{\{|f|>\gamma\}}\right\|_{\mathcal{M}^p_{q_2,r}} &\leq \left\|\gamma\chi_{\{|f|>\gamma\}}\right\|_{\mathcal{M}^p_{q_1,r}} \\
&\leq \sup_{\gamma>0}\left\|\gamma\chi_{\{|f|>\gamma\}}\right\|_{\mathcal{M}^p_{q_1,r}} \\
&= \|f\|_{w\mathcal{M}^p_{q_1,r}} \\
&< \infty.
\end{aligned}$$

Taking supremum over all $\gamma > 0$, we have

$$\sup_{\gamma>0}\left\|\gamma\chi_{\{|f|>\gamma\}}\right\|_{\mathcal{M}^p_{q_2,r}} = ||f||_{w\mathcal{M}^p_{q_2,r}} < \infty.$$

Hence, $w\mathcal{M}^p_{q_1,r} \subseteq w\mathcal{M}^p_{q_1,r}$. ∎

In Lemma 3.2, we obtain the inclusion $\mathcal{M}^p_{q,r} \subseteq w\mathcal{M}^p_{q,r}$. The following lemma is an inclusion result where the value of parameters q and r varies.

Lemma 3.5. *For* $1 \leq q_1 \leq q_2 \leq p < r_2 \leq r_1 \leq \infty$ *holds*

$$w\mathcal{M}^p_{q_2,r_2} \subseteq \mathcal{M}^p_{q_1,r_1}.$$

Proof. Let $f \in w\mathcal{M}^p_{q_2,r}$. Then

$$\begin{aligned}&\int_{Q_{vm}} |f(x)|^{q_1}\, dx\\ =&q_1 \int_0^\infty t^{q_1-1} \left|\{x \in Q_{vm} : |f(x)| > t\}\right| dt\\ \leq& |Q_{vm}|\, R^{q_1} + \frac{q_1}{q_2-q_1} |Q_{vm}|^{1-\frac{q_2}{p}} \left[|Q_{vm}|^{\frac{1}{p}-\frac{1}{q_2}}\, t\, |\{x \in Q_{vm} : |f(x)| > t\}|\right]^{q_2} R^{q_1-q_2}.\end{aligned}$$

If $y_{vm} = \left[|Q_{vm}|^{\frac{1}{p}-\frac{1}{q_2}}\, t\, |\{x \in Q_{vm} : |f(x)| > t\}|\right]$, we get

$$\int_{Q_{vm}} |f(x)|^{q_1}\, dx \leq |Q_{vm}|\, R^{q_1} + \frac{q_1}{q_2-q_1} |Q_{vm}|^{1-\frac{q_2}{p}} y_{vm}^{q_2} R^{q_1-q_2}.$$

By choosing $R = \left(\frac{q_1}{q_2-q_1}\right)^{\frac{1}{q_2}} |Q_{vm}|^{-\frac{1}{p}} y_{vm}$, it can be obtained

$$\int_{Q_{vm}} |f(x)|^{q_1}\, dx \leq 2 \cdot |Q_{vm}|^{1-\frac{q_1}{p}} \left(\frac{q_1}{q_2-q_1}\right)^{\frac{q_1}{q_2}} y_{vm}^{q_1}.$$

By taking q_1-root of both sides and dividing both sides by $|Q_{vm}|^{\frac{1}{q_1}-\frac{1}{p}}$, we get

$$\begin{aligned}|Q_{vm}|^{\frac{1}{p}-\frac{1}{q_1}} \left(\int_{Q_{vm}} |f(x)|^{q_1}\, dx\right)^{\frac{1}{q_1}} &\leq 2^{\frac{1}{q_1}} \cdot \left(\frac{q_1}{q_2-q_1}\right)^{\frac{1}{q_2}} y_{vm}\\ &= C \cdot |Q_{vm}|^{\frac{1}{p}-\frac{1}{q_2}} \left(t\, |\{x \in Q_{vm} : |f(x)| > t\}|\right).\end{aligned}$$

Since the above inequality is satisfied for each cube Q_{vm}, we obtain

$$\begin{aligned}&\|f\|_{\mathcal{M}^p_{q_1,r}}\\ =& \left\| |Q_{vm}|^{\frac{1}{p}-\frac{1}{q_1}} \left(\int_{Q_{vm}} |f(x)|^{q_1}\, dx\right)^{\frac{1}{q_1}} \right\|_{\ell^r}\\ \leq& \ C \left\| |Q_{vm}|^{\frac{1}{p}-\frac{1}{q_2}} \left(t\, |\{x \in Q_{vm} : |f(x)| > t\}|\right) \right\|_{\ell^r}\\ \leq& \ C\, \|f\|_{w\mathcal{M}^p_{q_2,r}}.\end{aligned}$$

Therefore, it is proven that $w\mathcal{M}^p_{q_2,r} \subseteq \mathcal{M}^p_{q_1,r}$. Based on the inclusion relationship in the sequence space ℓ^r we obtain $w\mathcal{M}^p_{q_2,r_2} \subseteq \mathcal{M}^p_{q_1,r_1}$ for $r_2 \leq r_1$. ■

The results of inclusion that have been obtained in weak Bourgain-Morrey spaces are similar to the results in weak Morrey spaces. Based on Lemmas 3.3, 3.4, and 3.5, we have the inclusions $\mathcal{M}^p_{q_2,r_2} \subseteq w\mathcal{M}^p_{q_2,r_2} \subseteq \mathcal{M}^p_{q_1,r_1}$ holds for $1 \leq q_1 < q_2 \leq p \leq r_2 \leq r_1 \leq \infty$.

4. Conclusion

In this research, we have some inclusion results of weak Bourgain-Morrey spaces. We introduce the definition of weak Bourgain-Morrey spaces by combining the definition of Bourgain-Morrey spaces in Definition 2.4 and the definition of weak Morrey spaces in Definition 2.1. Similar with inclusion properties of Bourgain-Morrey spaces, we have $w\mathcal{M}^p_{q,r_1} \subseteq w\mathcal{M}^p_{q,r_2}$ for $1 \le q \le p < r_1 \le r_2 \le \infty$ and $w\mathcal{M}^p_{q_1,r} \subseteq w\mathcal{M}^p_{q_2,r}$ for $1 \le q_2 \le q_1 \le p < r \le \infty$. For $1 \le q_1 < q_2 \le p \le r_2 \le r_1 \le \infty$, we got $w\mathcal{M}^p_{q_2,r_2} \subseteq \mathcal{M}^p_{q_1,r_1}$. Based on the results obtained above, we obtain

$$\mathcal{M}^p_{q_2,r_2} \subseteq w\mathcal{M}^p_{q_2,r_2} \subseteq \mathcal{M}^p_{q_1,r_1}.$$

for $1 \le q_1 < q_2 \le p \le r_2 \le r_1 \le \infty$.

5. Acknowledgement

This research is supported by FMIPA-ITB under FMIPA-PPMI-KK 2024 award no. 616AO/IT1.CO2/KU/2024.

References

[1] H. Gunawan, D. I. Hakim, M. Idris, Proper inclusions of Morrey spaces, *Glas. Mat. Ser.* III **53(73)**, no. 1, (2018) 143–151.

[2] H. Gunawan, D. I. Hakim, K. M. Limanta, A. A. Masta, Inclusion properties of generalized Morrey spaces, *Math. Nachr.* **290** (2017) 332–340.

[3] H. Gunawan, D. I. Hakim, E. Nakai, Y. Sawano, On inclusion relation between weak Morrey spaces and Morrey spaces, *Nonlinear Anal.* **168** (2018) 27–31.

[4] N. Hatano, T. Nogayama, Y. Sawano, D. I. Hakim, Bourgain–Morrey spaces and their applications to boundedness of operators, *J. Funct. Anal.* **284** (2023), Paper No. 109720, 52 pp.

[5] C. B. Morrey, On the solutions of quasi-linear elliptic partial differential equations, *Trans. Amer. Math. Soc.* **43** (1938) 126–166.

[6] Y. Sawano, G. Di Fazio, D. I. Hakim, *Morrey Spaces. Introduction and Applications to Integral Operators and PDE's*, Monographs and Research Notes in Mathematics, vol. 1, Chapman and Hall CRC Press, Boca Raton, FL, 2020.

[7] Y. Sawano, H. Tanaka, Morrey spaces for non–doubling measures, *Acta Math. Sinica* **21** (2005), no. 6, 1535–1544.

[8] M. O. I. P. Tarigan, R. Gunadi, D. I. Hakim, Ifronika, Inclusion of mixed Morrey spaces, *Adv. Oper. Theory* **8**, 16 (2023).