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Isoperimetric problems on n-sided prisms

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Abstract In a two-dimensional figure, the isoperimetric problem refers to finding a two-dimensional figure that will produce the largest area among several shapes with equal perimeter. This research extends the isoperimetric problem to finding three-dimensional shapes with maximum volume among those having equal surface area. Our main goal is to solve the isoperimetric problem for prisms with regular n-sided base, prisms with irregular n-sided base, and cylinders. In this research, the discussion is limited to prisms with regular and irregular bases. Our problem is equivalent to the problem of finding the smallest surface area of a given three-dimensional figure with the same volume. We will use a geometric approach in our proof. We will see the relationship between isoperimetric problems in two-dimensional figures and isoperimetric problems in three-dimensional figures. We obtain the results of the isoperimetric problem from two prisms with regular n-sided bases and a prism with regular m-sided bases with $n \leq m$, two prisms with regular n-sided bases and a prism with circular bases (cylinder), and two prisms with regular n-sided bases and a prism with irregular n-sided bases.

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1. Introduction

The term isoperimetric is a combination of iso and perimetric. Namely Iso means the same, and perimetric means perimeter. The isoperimetric problem in plane geometry involves examining several shapes with the same perimeter and then finding the shape with the largest area [4, 10]. Pappus of Alexandria (290–350 AD) proved a theorem that among plane shapes with the same perimeter, the area of a circle is larger than that of any polygon [3]. This is related to the Isoperimetric Theorem, which states that "A circle has the largest area among all plane shapes with the same perimeter." The resolution of the isoperimetric problem from the Isoperimetric Theorem indicates that a circle has the largest area among all other plane shapes with the same perimeter. There are more specific cases of the Isoperimetric Theorem [2, 4]. First, an equilateral triangle has the

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largest area among all triangles with the same perimeter. Second, a regular polygon has the largest area among all polygons with the same perimeter [5]. In simple terms, a specific case of the isoperimetric problem in a plane is determining the triangle with the maximum area among all triangles with a fixed perimeter [8]. Several theorems in the isoperimetric problem have not yet been proven in more detail. Therefore, more detailed and thorough proofs are needed, which will be discussed in this research.

The isoperimetric problem in this research will be discussed in greater detail and depth. In general, this study only discusses the isoperimetric problem in three-dimensional figures. The discussion is limited to prisms with regular bases and prisms with irregular bases. A prism is a three-dimensional shape bounded by two n-shaped sides that are parallel and congruent and the vertical sides are parallelograms. A regular prism is a prism whose sides are perpendicular to its base, with rectangular lateral faces. The aim of this research is to solve the Isoperimetric problem for three-dimensional figures of prisms. The method used is geometric proof. we will see the relationship between isoperimetric problems in two-dimensional figures and isoperimetric problems in three-dimensional figures.

2. Theoretical Study

The isoperimetric problem in a two-dimensional figure was then extended to the isoperimetric problem in a three-dimensional figure. Kazarinoff mentioned the isoperimetric problem in three-dimensional figures in his book titled Geometric Inequalities [5]. There are two equivalent statements: first, among all solids with the same surface area, a sphere has the largest volume; second, among all three-dimensional figures with the same volume, a sphere has the smallest surface area. Consider one of the statements that among all three-dimensional figures with the same surface area, the sphere has the largest volume. This strengthens the isoperimetric problem in three-dimensional figures, indicating that a sphere is the solid with the maximum volume among all three-dimensional figures with the same surface area. However, Kazarinoff did not explain in detail the solution to the isoperimetric problem in three-dimensional figures, and there is no proof for either of these statements.

Another statement mentioned by Kazarinoff regarding the isoperimetric problem in solid geometry pertains to prisms. The statement reads, "Among all rectangular prisms with the same volume, a cube has the smallest surface area."[5]. Unlike the previous two statements, which lack of proof, Kazarinoff proved this statement. The proof involves transforming the sides of an arbitrary rectangular prism into a square while maintaining the same surface area. As a result of this statement, among all rectangular prisms with the same surface area, a cube has the largest volume.

In another study, the proof of the isoperimetric problem on right prisms and oblique prisms with rectangular, right-angled triangle, and hexagonal bases was discussed using the most elementary applications of algebraic and trigonometric inequalities [9]. The results of the study showed that a right prism with a rectangular, right-angled triangle, or hexagonal base will have a larger volume than an oblique prism if both have the same surface area. Several cases of prisms have been previously solved through analytical calculations [1]. The order of maximum volumes of geometric shapes, if their surface areas are the same, from smallest to largest, is a prism with an equilateral triangular base, a prism with a square base, a prism with a regular n-sided base $n \geq 5$, and a cylinder.

The isoperimetric problem in three-dimensional figures mentioned by Kazarinoff is one of the intriguing topics that warrants a more detailed discussion. The isoperimetric problem in three-dimensional figures, particularly concerning prisms with triangular and rectangular bases, is a fascinating area of study. In this research, the isoperimetric problem can be expanded with the question: among three-dimensional figures with the same surface area, which solid will have the largest volume?; Here are the detailed problem statements in this research: Among all prisms with regular n-sided base and regular m-sided base, with $n \leq m$ and the same surface area, which prism has the largest volume?; Among all prisms with an n-sided base and a circular base (cylinder) with the same surface area, which prism has the largest volume?Among all prisms with regular n-sided bases and irregular n-sided bases with the same surface area, which prism has the largest volume?

Each of these problems is equivalent to the following problems. With the same volume, which three-dimensional figures has the smallest surface area?; Among all prisms with regular n-sided base and regular m-sided base, with $n \leq m$ and the same volume, which prism has the smallest surface area?; Among all prisms with an n-sided base and a circular base (cylinder) with the same volume, which prism has the smallest surface area? Among all prisms with regular n-sided base and irregular n-sided base with the same volume, which prism has the smallest surface area?

3. Main Results

The isoperimetric problem in a prism can be proven geometrically. It is known that "The problem of finding the largest volume of a given three-dimensional figure with the same surface area" is equivalent to "The problem of finding the smallest surface area of a given three-dimensional figure with the same volume". In this paper, we will compare two given three-dimensional figures with the same volume and determine which figure has the smallest surface area. Our proof uses the geometric approach as in the study carried out by Nogin [7]. First, we will consider a simple three-dimensional figure, namely a rectangular prism. Suppose we have a rectangular prism ABCD.EFGH. Choose one of its faces that is not a square and consider that face as its base. Construct the cube PQRD.STUH within the rectangular prism. Suppose the volume of the rectangular prism is equal to the volume of the cube, and the base area of the cube is equal to the base area of the prism (the area of the square equals the area of the rectangle). Consequently, the height of the cube is equal to the height of the prism. Illustration can be seen in Figure 1.

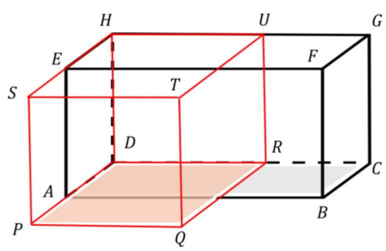

Figure 1. Rectangular Prism and Cube

Consider the bases of a cube and a rectangular prism. Since the area of the square equals the area of the rectangle, the perimeter of the square is smaller than that of the rectangle. This conclusion arises from the isoperimetric theorem in plane geometry, which states that among all quadrilaterals with the same area, a square has the smallest perimeter. Therefore, because the perimeter of a square is smaller than that of a rectangle, the surface area of a cube is smaller than that of a rectangular prism. Thus, the surface area of a cube is smaller than that of a rectangular prism when both shapes have equal volume.

First, we will discuss Lemma 3.1. Lemma 3.1 discusses the isoperimetric problem on flat shapes that will be used in Theorem 3.2. This lemma discusses comparing two polygons with different sides with the same area, which one has the smallest perimeter.

Lemma 3.1. *Suppose a regular n-gon and a regular m-gon with $n \leq m$. If both areas are the same, then the perimeter of the regular n-gon is greater than the perimeter of the regular m-gon.*

Proof. Let the perimeter of the n-gon is K and the area of the n-gon is $L = \frac{n}{2}r^2 \sin(\frac{2\pi}{n})$. Let the side lengths s, r be the radius of the circle containing the n-gon, and a is the apothem. $a^2 + (\frac{s}{2})^2 = r^2$ with $s = \frac{K}{n}$ dan $a = r\cos(\frac{\pi}{n})$, we get $\frac{K^2}{4n^2 \sin^2(\frac{\pi}{n})} = r^2$. Substitute r for the area of the n-gon, we get $L = \frac{K^2 \cos(\frac{\pi}{n})}{4n \sin(\frac{\pi}{n})}$. Let function $f(x) = \frac{\cos\frac{\pi}{x}}{x \sin\frac{\pi}{x}} = \frac{cot\frac{\pi}{x}}{x}$ with discrete variable $f(x)$, we will show that $f(x)$ is increasing. Obtained that $f'(x) = \frac{\pi - \frac{x}{2}\sin(\frac{2\pi}{n})}{x^3 \sin^2(\frac{\pi}{2})} \geq 0$, so $f(x)$ is increasing for $x \geq 3$. If the areas of the two shapes are the same, and $n \geq 3$, then we get that the perimeter of the regular n-gon is greater than the perimeter of the regular m-gon. ■

Our next theorem is Theorem 3.2, which compares two prisms with the same base area but different shapes.

Theorem 3.2. *Suppose two prisms with regular n-sided bases and a prism with regular m-sided bases with $n \leq m$. If the volumes of the two shapes are the same, then the surface area of a prism with a regular n-sided base is greater than the surface area of a prism with a regular m-sided base.*

Proof. Suppose two prisms with regular n-sided bases and a prism with regular m-sided bases with $n \leq m$. Suppose each side of the base of the two figures is n-sides and m-sides. Suppose the volumes of both prisms are the same. Suppose the base areas of both prisms are the same. (the area of a regular n-sided is equal to the area of a regular m-sided). As a result, the heights of both prisms are the same. Illustration can be seen in Figure 2. Consider that the two bases of the prism, based on Lemma 3.1, because the area of the n-gon is equal to the area of the m-gon, then the perimeter of the n-gon is greater than the perimeter of the m-gon.

Because the perimeter of the regular n-gon is greater than the perimeter of the regular m-gon, the surface area of the regular n-sided prism is consequently greater than the surface area of the regular m-sided prism. Let X be the surface area of a prism with a regular n-sided base and let Y be the surface area of a prism with a regular m-sided base. If the volumes of the two shapes are the same, then it will be true that $X \leq Y$.

Therefore, the surface area of the regular n-sided prism is greater than the surface area of the regular m-sided prism if the volumes of the two figures are the same. ■

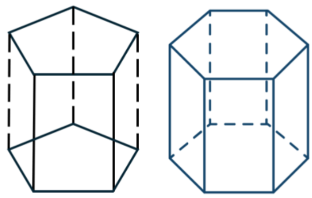

FIGURE 2. Pentagonal Base and Hexagonal Base Prism

Before discussing the next case, we need Lemma 3.3 first. Lemma 3.3 discusses comparing polygons with circles that have the same area, which one has the smallest circumference.

Lemma 3.3. *Suppose there is a regular n-sided polygon and a circle. If both areas are the same, then the perimeter of the regular n-sided polygon is greater than the circumference of the circle.*

Proof. Let the perimeter of the n-gon is K and the area of the n-gon is $L = \frac{n}{2}r^2\sin(\frac{2\pi}{n})$. Let the side lengths s, r be the radius of the circle containing the n-gon, and a is the apothem. Since $a^2 + (\frac{s}{2})^2 = r^2$ with $s = \frac{K}{n}$ and $a = r\cos(\frac{\pi}{n})$, we get $\frac{K^2}{4n^2\sin^2(\frac{\pi}{n})} = r^2$. Substitute r for the area of the n-gon, we get $K = \sqrt{\frac{L4n\sin(\frac{\pi}{n})}{\cos\frac{\pi}{n}}}$. Let function $f(p) = \frac{p\sin(\frac{\pi}{p})}{\cos(\frac{\pi}{p})}$ with discrete variables $f(p)$, we will show that $f(p)$ is bounded below. Let $p = \frac{\pi}{t}$ and $t = \frac{\pi}{p}$, with $p \geq 3$. Obtained that $f(\frac{\pi}{t}) = \pi\frac{\tan t}{t} > \pi \cdot 1 = \pi$, so $f(p)$ bounded below. If the areas of the two shapes are the same, then we get that the perimeter of the regular n-sided polygon is greater than the circumference of the circle. ■

Lemma 3.3 is also found in this source [2], which reads: "Of all plane figures with the same area, the circle has the smallest perimeter". Next is Theorem 3.7, which compares two prisms with polygons and circles that have the same base area.

Theorem 3.4. *Suppose there are two prisms, one with a base of a regular n-sided polygon and the other with a base of a circle. If the volumes of the two shapes are the same, then the surface area of the prism with the base of a regular n-sided polygon is greater than the surface area of the prism with the base of a circle.*

Proof. Suppose two prisms with regular n-sided bases and a prism with a circular base (cylinder). Suppose each side of the base of the two figures is an n-sided polygon and a circle. Suppose the volumes of both prisms are the same. Suppose the base areas of both prisms are the same. (the area of the regular n-sided polygon is equal to the area of the circle), As a result, the heights of both prisms are the same. Illustration can be seen in Figure 3. Consider that the two bases of the prism, based on Lemma 3.3, because the area of the n-sided polygon is equal to the area of the circle, the perimeter of the n-sided polygon must be greater than the circumference of the circle.

Because the perimeter of the n-sided polygon is greater than the circumference of the circle, the surface area of the prism with the n-sided polygon base is greater than the surface area of the prism with the circular base (cylinder). Suppose P is the surface area of a prism with a regular n-sided polygon and Q is the surface area of a prism with a

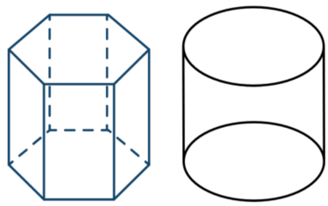

FIGURE 3. Hexagonal Base Prism and Cylinder

circular base (cylinder). If the volumes of the two shapes are the same, then it will be true that $P \leq Q$. Therefore, the surface area of the prism with a regular n-sided polygon base is greater than the surface area of the prism with a circular base (cylinder) if the volumes of the two figures are the same. ∎

The next case is Lemma 3.5. Lemma 3.5 discusses polygons in a circle.

Lemma 3.5. *Among all n-gons inscribed in a given circle, the regular n-gon has the smallest perimeter.*

Proof. Suppose $n > 4$. Suppose there are n points, namely $B_1, B_2, \ldots, B_n$, which lie on circle C with center O and radius r. Let B be a n-gon whose vertices are points $B_1, B_2, \ldots, B_n$. Let A be a regular n-gon whose vertices are points $A_1, A_2, \ldots, A_n$ lying on circle C. If A and B have the same area, then the perimeter of B is greater than the perimeter of A.

Construction of the n-gon B with each vertex, namely $B_1, B_2, \ldots, B_n$, which lies on circle C. Let β_i be the angle formed from lines B_iO and B_jO, $1 \leq i \leq n, 1 \leq j \leq n, i \neq j, x_i$ is the i-th side of B and K is the perimeter of B. We get $x_i = 2r\sin(\frac{\beta_i}{2})$ with $0 < \beta_i < \pi$. The circumference pf B is $K_B = \Sigma_{i=1}^n x_i = 2r\Sigma_{i=1}^n \sin(\frac{\beta_i}{2})$ with radius of C being $r = \frac{K_B}{2\Sigma_{i=1}^n \sin(\frac{\beta_i}{2})}$. The area of the n-gon B is $L = \frac{r^2}{2}\Sigma_{i=1}^n \sin(\beta_i)$, subtitute r to obtain $K_B = \sqrt{\frac{8L(\Sigma_{i=1}^n \sin(\frac{\beta_i}{2}))}{(\Sigma_{i=1}^n \sin(\beta_i))}}$.

Let β_i be the angle formed formed from lines A_iO and A_jO, $1 \leq i \leq n, 1 \leq j \leq n, i \neq j$, then $\beta_i = \frac{2\pi}{n}$. For this reason, the area of n-side A is $L = \frac{r^2}{2}\Sigma_{i=1}^n \sin(\frac{2\pi}{n})$, subtitute r to obtain $K_A = \sqrt{\frac{8Ln(\sin(\frac{\pi}{n}))^2}{\sin(\frac{2\pi}{n})}}$. Based on Jensen's inequality, $\sin(\frac{1}{n}\Sigma_{i=1}^n \beta_i) \geq \frac{1}{n}\Sigma_{i=1}^n \beta_i$. We get $\sqrt{\frac{8Ln(\sin(\frac{\pi}{n}))^2}{\sin(\frac{2\pi}{n})}} \leq \sqrt{\frac{8L(\Sigma_{i=1}^n \sin(\frac{\beta_i}{2}))}{(\Sigma_{i=1}^n \sin(\beta_i))}}$, or $K_A \leq K_B$.

Therefore, if the area of the two shapes is the same, then the perimeter of the regular n-gon is smaller than the perimeter of the irregular n-gon. Of all n-gons inscribed in a given circle, the regular n-gon has the smallest perimeter. ∎

From [5] and [6], the previous Lemma 3.5 is expanded again to obtain the following lemma. "Among all the n-gon with the same perimeter, the regular n-gon has the largest area" equivalent to "Among all n-gons with the same area, a regular n-gon has the smallest perimeter".

Lemma 3.6. *Among all n-gons with the same area, a regular n-gon has the smallest perimeter.*

Lemma 3.5 and Lemma 3.6 are also found in this source [5], which reads: "Of all n-gons inscribed in a given circle, the regular n-gon has the smallest perimeter". Next is Theorem 3.7, which compares two prisms with regular and irregular polygon bases.

Theorem 3.7. *Suppose two prisms with regular n-sided bases and a prism with irregular n-sided bases. If the volumes of the two shapes are the same, then the surface area of a prism with an irregular n-sided base is greater than the surface area of a prism with a regular n-sided base.*

Proof. Suppose two prisms with regular n-sided bases and a prism with irregular n-sided bases. Suppose each side of the base of the two figures is a regular n-sided and an irregular n-sided. Suppose the volumes of both prisms are the same. Suppose the base areas of both prisms are the same. (the area of a regular n-sided is the same as the area of an irregular n-sided), As a result, the heights of both prisms are the same. Illustration can be seen in Figure 4. Consider that the two bases of the shape, based on Lemma 3.6,

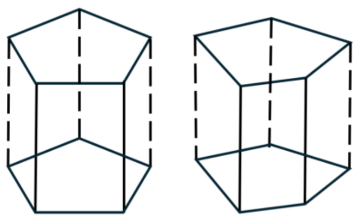

FIGURE 4. Regular and Irregular Pentagonal Base Prisms

because the area of the regular n-side is equal to the area of the irregular n-side, then the perimeter of the irregular n-side is greater than the perimeter of the regular n-side.

Because the perimeter of an irregular n-gon is greater than the perimeter of a regular n-gon, the surface area of a prism with an irregular n-sided base is greater than the surface area of a prism with a regular n-sided base. Suppose A is the surface area of a prism with an irregular n-sided base and B is the surface area of a prism with a regular n-sided base. If the volumes of the two shapes are the same, then it will be true that $A \leq B$.

Therefore, the surface area of a prism with an irregular n-sided base is greater than the surface area of a prism with a regular n-sided base if the volumes of the two figures are the same. ∎

4. Concluding Remarks

Suppose two prisms with regular n-sided bases and a prism with regular m-sided bases with $n \leq m$. If the volumes of the two shapes are the same, then the surface area of a prism with a regular n-sided base is greater than the surface area of a prism with a regular m-sided base. Suppose there are two prisms, one with a base of a regular n-sided polygon and the other with a base of a circle. If the volumes of the two shapes are the same, then the surface area of the prism with the base of a regular n-sided polygon is greater than the surface area of the prism with the base of a circle. Suppose two prisms with regular n-sided bases and a prism with irregular n-sided bases. If the volumes of the two shapes are the same, then the surface area of a prism with an irregular n-sided base is greater than the surface area of a prism with a regular n-sided base.

Suggestions for further research are Isoperimetric problems on pyramids with regular n-sided bases and pyramids with irregular n-sided bases. Some cases of the pyramid have been previously solved through analytical calculations. The order of maximum volume of geometric figures, if the surface area is the same, from smallest to largest, is a pyramid with an equilateral triangular base, a pyramid with a square base, a pyramid with a regular n-sided base $n \geq 5$, and a cone. However, the solution has not been found, we cannot prove it geometrically for the isoperimetric problem in the pyramid.

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References

[1] A. M. Ali, D. I. Hakim, Eksplorasi Masalah Isoperimetrik pada Bangun Ruang, *Euler: Jurnal Ilmiah Matematika, Sains dan Teknologi* **12** (2024) 16–25.
[2] V. Blåsjö, The Isoperimetric Problem, *The American Mathematical Monthly* **112** (2005) 526.
[3] I. Bulmer-Thomas, *Selections illustrating the history of Greek mathematics*, Cambridge, Mass.: Harvard University Press, **2**, 1939.
[4] R. F. Demar, A Simple Approach to Isoperimetric Problems in the Plane, *Mathematics Magazine* **48** (1975) 1.
[5] N. D. Kazarinoff, *Geometric inequalities*, Yale University, 1961.
[6] I. Niven, *Maxima and minima without calculus*, Cambridge University Press, **6**, 1981.
[7] M. Nogin, It is not a coincidence! On curious patterns in calculus optimization problems, (2016) *arXiv preprint arXiv:1603.08542.*
[8] J. A. Rizcallah, Isoperimetric Triangles, *The Mathematics Teacher* **111** (2017) 70–74.
[9] A. Setiawan, D. I. Hakim, O. Neswan, Perluasan Masalah Isoperimetrik pada Bangun Ruang, *Jambura Journal of Mathematics* **5** (2023) 380–403.
[10] J. Tanton, The Isoperimetric Problem, *Math Horizons* **10** (2003) 23–26.