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Rate of convergence of Kantorovich operator sequences near $L^1([0,1])$

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Abstract The study of the rate of convergence of Kantorovich operator sequences has predominantly focused on the L^p spaces for $1 < p < \infty$, yet the behaviour near $L^1([0,1])$ remains less understood, particularly as p approaches 1. To bridge this gap, we investigate the rate of convergence within the framework of the grand Lebesgue spaces $L^{p)}([0,1])$, which encompass all $L^p([0,1])$ spaces for $1 < p < \infty$ but remain a subset of $L^1([0,1])$.

Our approach leverages the intrinsic properties of $L^{p)}([0,1])$ to derive new results on the convergence rate of Kantorovich operator sequences. Specifically, our objective is to demonstrate that Kantorovich operators exhibit a significant rate of convergence within this broader context, thereby providing insights applicable to the boundary behavior as $p \to 1$.

We will then apply these findings to α-Hölder continuous functions to further understand the convergence rate of Kantorovich operator sequences in these settings. This combined approach suggests that functions with derivatives in $L^{p)}([0,1])$ exhibit specific convergence rates under Kantorovich operators.

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1. Introduction

Kantorovich operators are widely recognized for their utility in approximating functions, particularly because of their applicability to integrable functions. After its introduction by Kantorovich as an extension of Bernstein polynomials, these operators have proven to be a valuable tool in approximation theory, especially in contexts where continuity cannot be assumed. Unlike Bernstein polynomials, which are limited to continuous functions, Kantorovich operators can approximate functions within broader function spaces, such as $L^p([0,1])$ spaces, making them a more versatile tool for analysis [7]. Although Kantorovich operators are uniformly bounded for $1 \le p \le \infty$, this sequence of operators is not convergent for $p = \infty$ (see [9]).

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The study of the convergence of Kantorovich operators in $L^p([0,1])$ spaces, for $1 \leq p < \infty$, has shown that these operators converge back to the function they approximate, with a rate of convergence that has been extensively analyzed [8]. However, the result about the rate of convergence of Kantorovich operators on $L^1([0,1])$ is not clear as the convergence in $L^p([0,1])$ for $p > 1$. One of the reasons for this lack of property is the unboundedness of the Hardy-Littlewood maximal operator on $L^1([0,1])$.

To address these challenges, grand Lebesgue spaces $L^{p)}([0,1])$ provide a useful framework, acting as an intermediate space between $L^p([0,1])$ and $L^1([0,1])$ [6]. These spaces capture the finer nuances of function behavior near $L^1([0,1])$ and allow for a more refined analysis of operator convergence [3]. The use of grand Lebesgue spaces in the study of Kantorovich operators is particularly advantageous because, although the rate of convergence is generally slower compared to $L^p([0,1])$ spaces, the broader scope of functions included in $L^{p)}([0,1])$ makes this framework essential for understanding convergence near $L^1([0,1])$ [1]. We refer the reader to [2] and [10] for uniform boundedness and the convergence of Kantorovich operators on other function spaces.

In this paper, we investigate the rate of convergence of Kantorovich operators within grand Lebesgue spaces $L^{p)}([0,1])$. Additionally, we explore the implications of these results for α-Hölder continuous functions, further elucidating the behavior of Kantorovich operators in these broader spaces. We demonstrate that while the rate of convergence in $L^{p)}([0,1])$ spaces is slower than in classical $L^p([0,1])$ spaces, this trade-off is justified by the inclusion of a wider class of functions.

2. Preliminaries

In this section, we introduce essential definitions, theorems, and results that form the foundation for the main results of this paper. First, we define the Kantorovich operators, which play a central role in approximation theory.

Definition 2.1 (Kantorovich operators [7])**.** Let $n \in \mathbb{N}$ and f be an integrable function on $[0,1]$. The Kantorovich operator of order n is defined as

$$K_n f(x) := \sum_{k=0}^{n} (n+1) \left[\int_{I_{n,k}} f(t)\, dt \right] b_{n,k}(x) \qquad , x \in [0,1]$$

where

$$b_{n,k}(x) := \binom{n}{k} x^k (1-x)^{n-k}$$

and

$$I_{n,k} := \begin{cases} \left[\frac{k}{n+1}, \frac{k+1}{n+1}\right), & k \neq n \\ \left[\frac{k}{n+1}, \frac{k+1}{n+1}\right], & k = n. \end{cases}$$

In analyzing the behavior of Kantorovich operators, it is often necessary to consider the Hardy-Littlewood maximal operator. This operator provides a way to control the size of a function and is crucial for establishing boundedness properties.

Definition 2.2 (Hardy-Littlewood Maximal Operator [5]). If f is integrable on $[0,1]$, the maximal function of f is defined as

$$Mf(x) := \sup_{x\in[a,b]} \frac{1}{b-a}\int_a^b |f(y)|\,dy, \qquad x\in[0,1],$$

where the supremum is taken over all closed intervals in $[0,1]$ that contain x.

We now introduce the concept of grand Lebesgue spaces, which provide a broader context for analyzing the behavior of Kantorovich operators near $L^1([0,1])$.

Definition 2.3 (Grand Lebesgue spaces [6]). The grand Lebesgue space on $[0,1]$, denoted by $L^{p)}([0,1])$, for $1<p<\infty$, is the space of all measurable functions $f:[0,1]\to\mathbb{R}$ that satisfy

$$\|f\|_{L^{p)}([0,1])} := \sup_{0<\varepsilon<p-1}\left(\varepsilon\int_0^1 |f(x)|^{p-\varepsilon}\,dx\right)^{\frac{1}{p-\varepsilon}} < \infty.$$

We recall the following inclusion result.

Theorem 2.4 ([3]). *It is known that for $1<p<\infty$, we have*

$$L^p([0,1]) \subset L^{p)}([0,1]) \subset L^1([0,1]).$$

The space $L^{p)}([0,1])$ is larger than $L^p([0,1])$ but smaller than $L^1([0,1])$, making it an intermediate space that encompasses functions that may not belong to $L^p([0,1])$ but still exhibit useful integrability properties.

A key result in approximation theory is the rate of convergence of Kantorovich operator sequences in $L^p([0,1])$ spaces.

Theorem 2.5 ([8]). *Let $1<p<\infty$. If $f,f',f''\in L^p([0,1])$, then there exists a constant $C>0$ such that for every $n\in\mathbb{N}$,*

$$\|K_nf-f\|_{L^p([0,1])} \le C\frac{\|f'\|_{L^p([0,1])}+\|f''\|_{L^p([0,1])}}{n}.$$

The concept of α-Hölder continuous functions can be used to further investigate the rate of convergence of Kantorovich operator sequences.

Definition 2.6 (α-Hölder Continuous Functions). Let $0<\alpha\le 1$. The function $f:[0,1]\to\mathbb{R}$ is called an α-Hölder continuous function on $[0,1]$ if there exists a constant $C\ge 0$ such that

$$|f(x)-f(y)| \le C|x-y|^\alpha \quad ,\forall x,y\in[0,1].$$

The seminorm of f is given by

$$|f|_{C^{0,\alpha}([0,1])} := \sup_{\substack{x,y\in[0,1]\\ x\neq y}} \frac{|f(x)-f(y)|}{|x-y|^\alpha}.$$

Next, we present some additional key results in $L^p([0,1])$ spaces that will be essential in extending our analysis to grand Lebesgue spaces.

Theorem 2.7 ([1]). *Let $1<p<\infty$ and $f\in C^{0,\alpha}([0,1])$ with $0<\alpha\le 1$. Then, for every $n\in\mathbb{N}$,*

$$\|K_nf-f\|_{L^p([0,1])} \le \left(\frac{1}{4n}\right)^{\frac{\alpha}{2}} |f|_{C^{0,\alpha}([0,1])}.$$

Theorem 2.7 follows from the following lemma by applying it to a function f with α-Hölder continuity and then integrating it over $[0,1]$ in the L^p-norm.

Lemma 2.8 ([1]). *Suppose* $0 < \alpha \leq 1$. *For every* $n \in \mathbb{N}$ *and* $x \in [0,1]$, *we have*

$$K_n(|\cdot - x|^\alpha)(x) \leq \left(\frac{1}{4n}\right)^{\frac{\alpha}{2}}.$$

We recall the following embedding between Sobolev spaces and Holder continuous classes.

Lemma 2.9 ([1]). *Let* $1 < p < \infty$. *If* f *is a function such that* $f' \in L^p([0,1])$, *then*

$$|f|_{C^{0,1-1/p}([0,1])} \leq \|f'\|_{L^p([0,1])}.$$

Therefore, Theorem 2.7 implies the following corollary.

Corollary 2.10 ([1]). *Let* $1 < p < \infty$ *and* $f, f' \in L^p([0,1])$. *Then for every* $n \in \mathbb{N}$,

$$\|K_n f - f\|_{L^p([0,1])} \leq \left(\frac{1}{4n}\right)^{\frac{1}{2}\left(1-\frac{1}{p}\right)} \|f'\|_{L^p([0,1])}.$$

3. Main Results

Building upon the foundations laid in the preliminaries, we now present our main results.

Theorem 3.1 (Rate of convergence of Kantorovich operator sequences in $L^{p)}$ spaces). *Let* $1 < p < \infty$. *If* $f, f', f'' \in L^{p)}([0,1])$ *then there exists a constant* $C > 0$ *such that for every* $n \in \mathbb{N}$,

$$\|K_n f - f\|_{L^{p)}([0,1])} \leq C \frac{\|f'\|_{L^{p)}([0,1])} + \|f''\|_{L^{p)}([0,1])}}{n}.$$

Proof. Consider a function $f \in L^{p)}([0,1])$ such that $f', f'' \in L^{p)}([0,1])$. Similar to the proof of Theorem 2.5 as shown in [8], we can get

$$|K_n f(x) - f(x)| \leq \frac{|f'(x)|}{4n} + \frac{11 M f''(x)}{24n}.$$

Let $0 < \varepsilon < p-1$. Using Minkowski's inequality and then multiplying both side by $\varepsilon^{\frac{1}{p-\varepsilon}}$, we get

$$\left(\varepsilon \int_0^1 |K_n f(t) - f(t)|^{p-\varepsilon}\, dt\right)^{\frac{1}{p-\varepsilon}} \leq \frac{1}{4n} \left(\varepsilon \int_0^1 |f'(t)|^{p-\varepsilon}\, dt\right)^{\frac{1}{p-\varepsilon}} + \frac{11}{24n} \left(\varepsilon \int_0^1 (M f''(t))^{p-\varepsilon}\, dt\right)^{\frac{1}{p-\varepsilon}}.$$

Taking the supremum over $0 < \varepsilon < p-1$ on both sides, we obtain

$$\|K_n f - f\|_{L^{p)}([0,1])} \leq \frac{1}{4n} \|f'\|_{L^{p)}([0,1])} + \frac{11}{24n} \|M f''\|_{L^{p)}([0,1])}.$$

Using Corollary 2.3 in [3], we conclude that

$$\|K_n f - f\|_{L^{p)}([0,1])} \leq \frac{C}{n} (\|f'\|_{L^{p)}([0,1])} + \|f''\|_{L^{p)}([0,1])}).$$

■

Theorem 3.2. *Let $1 < p < \infty$ and $0 < \varepsilon_0 < p-1$. Define $\alpha_0 := 1 - \frac{1}{p-\varepsilon_0}$. If $f' \in L^{p)}([0,1])$, then*

$$|f|_{C^{0,\alpha_0}([0,1])} \leq \varepsilon_0^{-\frac{1}{p-\varepsilon_0}} \|f'\|_{L^{p)}([0,1])}.$$

Proof. Let $1 < p < \infty$ and $0 < \varepsilon_0 < p-1$. Clearly, $0 < 1 - \frac{1}{p-\varepsilon_0} < 1$. Define $\alpha_0 := 1 - \frac{1}{p-\varepsilon_0}$. Consider any function f such that $f' \in L^{p)}([0,1])$. Let $x, y \in [0,1]$. Without loss of generality, assume $x < y$.

Using Hölder's inequality, we obtain

$$\begin{aligned} |f(x) - f(y)| &\leq \int_x^y |f'(t)|\, dt \\ &\leq \left(\int_x^y |f'(t)|^{p-\varepsilon_0}\, dt\right)^{\frac{1}{p-\varepsilon_0}} \left(\int_x^y dt\right)^{1-\frac{1}{p-\varepsilon_0}} \\ &= \varepsilon_0^{-\frac{1}{p-\varepsilon_0}} \left(\varepsilon_0 \int_x^y |f'(t)|^{p-\varepsilon_0}\, dt\right)^{\frac{1}{p-\varepsilon_0}} (y-x)^{1-\frac{1}{p-\varepsilon_0}}. \end{aligned}$$

Dividing both sides by $(y-x)^{\alpha_0}$, we get

$$\frac{|f(x) - f(y)|}{(y-x)^{\alpha_0}} \leq \varepsilon_0^{-\frac{1}{p-\varepsilon_0}} \left(\varepsilon_0 \int_x^y |f'(t)|^{p-\varepsilon_0}\, dt\right)^{\frac{1}{p-\varepsilon_0}}.$$

Since x and y are arbitrary points in $[0,1]$, the previous inequality holds for the α-Hölder seminorm of f,

$$|f|_{C^{0,\alpha_0}([0,1])} \leq \varepsilon_0^{-\frac{1}{p-\varepsilon_0}} \left(\varepsilon_0 \int_x^y |f'(t)|^{p-\varepsilon_0}\, dt\right)^{\frac{1}{p-\varepsilon_0}}.$$

Then, since $0 < \varepsilon_0 < p-1$ and $f' \in L^{p)}([0,1])$,

$$|f|_{C^{0,\alpha_0}([0,1])} \leq \varepsilon_0^{-\frac{1}{p-\varepsilon_0}} \|f'\|_{L^{p)}([0,1])}.$$

∎

Theorem 3.3. *Let $1 < p < \infty$ and $0 < \varepsilon_0 < p-1$. If $f, f' \in L^{p)}([0,1])$, then for every $n \in \mathbb{N}$, we have*

$$\|K_n f - f\|_{L^{p)}([0,1])} \leq \frac{C}{(4n)^{\alpha_0/2}} \varepsilon_0^{-\frac{1}{p-\varepsilon_0}} \|f'\|_{L^{p)}([0,1])},$$

where $\alpha_0 := 1 - \frac{1}{p-\varepsilon_0}$ and $C = \max\{1, p-1\}$.

Proof. Let $1 < p < \infty$ and $0 < \varepsilon_0 < p-1$. Consider a function f such that $f, f' \in L^{p)}([0,1])$. Let $n \in \mathbb{N}$ and $x \in [0,1]$. Then,

$$\begin{aligned} |K_n f(x) - f(x)| &= \left|\sum_{k=0}^{n} (n+1) \left[\int_{I_{n,k}} f(t)\, dt\right] b_{n,k}(x) - f(x)\right| \\ &\leq \sum_{k=0}^{n} (n+1) \left[\int_{I_k} |f(t) - f(x)|\, dt\right] b_{n,k}(x). \end{aligned}$$

Since $t, x \in [0,1]$, it follows that

$$|f(t) - f(x)| \leq |f|_{C^{0,\alpha_0}([0,1])} |t-x|^{\alpha_0}.$$

Therefore,

$$
\begin{aligned}
|K_n f(x) - f(x)| &\leq \sum_{k=0}^{n} (n+1) \left[\int_{I_k} |f|_{C^{0,\alpha_0}([0,1])} |t-x|^{\alpha_0} \, dt \right] b_{n,k}(x) \\
&= \sum_{k=0}^{n} (n+1) \left[\int_{I_k} |t-x|^{\alpha_0} \, dt \right] b_{n,k}(x) \cdot |f|_{C^{0,\alpha_0}([0,1])} \\
&= K_n(|\cdot - x|^{\alpha_0})(x) \cdot |f|_{C^{0,\alpha_0}([0,1])}.
\end{aligned}
$$

Using Lemma 3.5 on [1] which states that

$$
K_n(|\cdot - x|^{\alpha_0})(x) \leq \left(\frac{1}{4n} \right)^{\frac{\alpha_0}{2}}
$$

for every $n \in \mathbb{N}$ and $x \in [0,1]$, we get

$$
|K_n f(x) - f(x)| \leq \left(\frac{1}{4n} \right)^{\frac{\alpha_0}{2}} |f|_{C^{0,\alpha_0}([0,1])}.
$$

Taking the $L^{p)}[0,1]$ norm from both sides, we obtain

$$
\begin{aligned}
\|K_n f - f\|_{L^{p)}([0,1])} &\leq \left\| \left(\frac{1}{4n} \right)^{\frac{\alpha_0}{2}} |f|_{C^{0,\alpha_0}([0,1])} \right\|_{L^{p)}([0,1])} \\
&\leq \left(\frac{1}{4n} \right)^{\frac{\alpha_0}{2}} |f|_{C^{0,\alpha_0}([0,1])} \|1\|_{L^{p)}([0,1])}.
\end{aligned}
$$

According to [4], $\|1\|_{L^{p)}([0,1])} = \sup_{\varepsilon} \varepsilon^{\frac{1}{p-\varepsilon}} \leq \max\{1, p-1\}$ then by implementing Theorem 3.2 we can write the previous inequality as

$$
\|K_n f - f\|_{L^{p)}([0,1])} \leq \frac{C}{(4n)^{\alpha_0/2}} \varepsilon_0^{-\frac{1}{p-\varepsilon_0}} \|f'\|_{L^{p)}([0,1])},
$$

where $\alpha_0 := 1 - \frac{1}{p-\varepsilon_0}$ and $C = \max\{1, p-1\}$. ∎

4. Concluding Remarks

From Corollary 2.10 and Theorem 3.3, we know that

$$
\|K_n f - f\|_{L^p([0,1])} = O\left(n^{-\frac{1}{2}\left(1-\frac{1}{p}\right)} \right)
$$

while

$$
\|K_n f - f\|_{L^{p)}([0,1])} = O\left(n^{-\frac{1}{2}\left(1-\frac{1}{p-\varepsilon_0}\right)} \right).
$$

It is clear that the sequence of Kantorovich operators $K_n f$ approaches f at a slower rate in $L^{p)}([0,1])$ spaces compared to in $L^p([0,1])$. However, this slower rate of convergence comes with a significant advantage: the space $L^{p)}([0,1])$ encompasses a broader class of functions than $L^p([0,1])$. This means that while the convergence is slower, it applies to a more extensive set of functions, offering a more flexible and general framework for approximation.

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