Hilbert Journal of Mathematical Analysis
Volume 3 Number 2 (2025)
Pages 034–059

https://hilbertjma.org

ISSN 2985-7619

A new integral inequality depending on the Lobachevskii function

Christophe Chesneau[1,*]

[1] *Department of Mathematics, LMNO, University of Caen-Normandie, 14032 Caen, France.*
e-mail: christophe.chesneau@gmail.com

Abstract This paper investigates a new logarithmic variant of the Hilbert integral inequality, beyond the standard homogeneous assumption. To this end, several theorems and propositions are proved, each of which gives new integral inequalities of independent interest. Remarkably, the Lobachevskii function appears quite naturally in some proofs, and forms an important part of the upper bound obtained. Several sharp examples are developed and discussed. A brief overview of the Lobachevskii function is given in the appendix.

MSC: 47A07; 26D15

Keywords: Hilbert integral inequality; Lobachevskii function; logarithmic function; integral inequalities

Received: 26-04-2025 / Accepted: 03-11-2025 / Published: 08-12-2025
DOI: https://doi.org/10.62918/hjma.v3i2.38

1. Introduction

The Hilbert integral inequality is a fundamental result in mathematical analysis and functional analysis. It provides bounds on the integral of the product of functions. It is particularly useful in studying various properties of function spaces. Formally, the Hilbert integral inequality can be stated as follows: For any quadratic integrable functions $f, g : (0,+\infty) \mapsto (0,+\infty)$, we have

$$\int_0^{+\infty}\int_0^{+\infty} \frac{1}{x+y} f(x)g(y)dxdy \leq \pi \sqrt{\int_0^{+\infty} f^2(x)dx}\sqrt{\int_0^{+\infty} g^2(x)dx},$$

(hereafter, without loss of generality, the point 0 can be included in the definition of f and g). Therefore, we have an upper bound depending on a constant, i.e., π, and the L_2 norms of f and g. In fact, the inequality is strict and the constant π cannot be improved. For more details, we refer to [6, 14]. This inequality has been the subject of extensive research and numerous generalizations. For a comprehensive overview, see [1, 3, 4, 10–12, 15]. In addition, the survey in [2] provides a valuable perspective on some of these advances.

*Corresponding author.

Published by Komunitas Analisis Matematika Indonesia.

Some logarithmic variants of the Hilbert integral inequality have been established in the literature, in particular. The most notable of these, which are free of adjustable parameters, are presented below.

- The logarithmic integral inequality in [6, Formula 342] can be formalized as follows:

$$\int_0^{+\infty}\int_0^{+\infty}\frac{1}{x-y}\log\left(\frac{x}{y}\right)f(x)g(y)dxdy \le \pi^2\sqrt{\int_0^{+\infty}f^2(x)dx}\sqrt{\int_0^{+\infty}g^2(x)dx}.$$

- In [13], the following logarithmic integral inequality is demonstrated:

$$\int_0^{+\infty}\int_0^{+\infty}\frac{1}{\max(x,y)}\left|\log\left(\frac{x}{y}\right)\right|f(x)g(y)dxdy \le 8\sqrt{\int_0^{+\infty}f^2(x)dx}\sqrt{\int_0^{+\infty}g^2(x)dx}.$$

- Another variant is given in [7], and it states that

$$\int_0^{+\infty}\int_0^{+\infty}\frac{1}{x+y+|x-y|}\left|\log\left(\frac{x}{y}\right)\right|f(x)g(y)dxdy \le 4\sqrt{\int_0^{+\infty}f^2(x)dx}\sqrt{\int_0^{+\infty}g^2(x)dx}.$$

- In [3], the following logarithmic integral inequality is established:

$$\int_0^{+\infty}\int_0^{+\infty}\frac{1}{x+y}\log\left(1+\frac{x}{y}\right)f(x)g(y)dxdy \le 2\pi\log(2)\sqrt{\int_0^{+\infty}f^2(x)dx}\sqrt{\int_0^{+\infty}g^2(x)dx}.$$

In this paper, we investigate a new logarithmic variant of the Hilbert integral inequality. It is based on the following double integral:

$$\int_0^{+\infty}\int_0^{+\infty}\frac{1}{x+y}\left|\log\left(\frac{1}{x+y}\right)\right|f(x)g(y)dxdy. \tag{1.1}$$

Therefore, it depends on the following kernel function:

$$K(x,y) := \frac{1}{x+y}\left|\log\left(\frac{1}{x+y}\right)\right|.$$

This kernel function has the feature to be non-homogeneous i.e., there does not exist a $\gamma > 0$ such that, for any $x, y > 0$ and $\lambda > 0$, we have $K(\lambda x, \lambda y) = \lambda^{-\gamma}K(x, y)$, i.e.,

$$\frac{1}{\lambda}\left[\frac{1}{x+y}\left|\log\left(\frac{1}{x+y}\right)-\log(\lambda)\right|\right] = \lambda^{-\gamma}\frac{1}{x+y}\left|\log\left(\frac{1}{x+y}\right)\right|.$$

This non-homogeneity is singular in the sense that it is mainly due to the term $\log(\lambda)$, which is complexly integrated into an absolute value operator.

To our knowledge, upper bounds for the double integral in (1.1) have not been considered in the literature. In particular, the treatment of the corresponding special kernel function is challenging from a mathematical point of view. In this paper, we study this aspect by means of two intermediate theorems (or “sub-theorems”), which can be taken independently. The main result can be obtained by combining them. In the first theorem, we focus on the following double integral:

$$\int\int_{\{(x,y)\in(0,+\infty)^2;\ x+y\leq 1\}} \frac{1}{x+y}\log\left(\frac{1}{x+y}\right) f(x)g(y)dxdy.$$

It is characterized by a non-separable integration domain. Based on this, we prove an elegant logarithmic integral inequality using probabilistic tools. Several statements are made that extend the scope of this result. In particular, we discuss how adjustable parameters can be introduced into the obtained inequalities in a sharp way.

In the second theorem, a complementary logarithmic integral inequality is proved. It is based on the following double integral:

$$\int\int_{\{(x,y)\in(0,+\infty)^2;\ x+y>1\}} \frac{1}{x+y}\log\left(\frac{1}{x+y}\right) f(x)g(y)dxdy.$$

In the proof, we adapt the polar change of variables strategy elaborated in [9] to find a sharp upper bound for it. It is interesting to note that, unlike the variants presented earlier, the Lobachevskii function introduced by N. I. Lobachevskii in 1829 (see [8]) turned out to be an important analytical tool. In particular, it directly affects the L_2 norms of f and g of the upper bound obtained. More precisely, it takes the following form:

$$\upsilon\sqrt{\int_0^{+\infty}\phi(x)f^2(x)dx}\sqrt{\int_0^{+\infty}\phi(x)g^2(x)dx},$$

where υ is a simple constant and ϕ is a function depending on the Lobachevskii function. We reach our objective by combining the results of the two theorems. An alternative proof giving a more conventional form of the upper bound is also given, still using the Lobachevskii function.

The plan for the rest of the paper is as follows: Section 2 is devoted to the first theorem and some results derived from it. Section 3 is the analogue of the second theorem. Section 4 examines the main logarithmic integral inequalities based on the previous results. A conclusion is given in Section 5. A brief overview of the Lobachevskii function and its properties is given in the appendix.

2. First theorem and derived results

2.1. First logarithmic integral inequality

The theorem below considers a new logarithmic integral inequality which has the property of depending on an non-separable integration domain (corresponding to the "unit triangle").

Theorem 2.1. *Let $f, g : (0, 1) \mapsto (0, +\infty)$ be two functions. Assuming that the integrals involved converge, we have*

$$\int\int_{\{(x,y)\in(0,+\infty)^2;\ x+y\leq 1\}} \frac{1}{x+y}\log\left(\frac{1}{x+y}\right) f(x)g(y)dxdy$$
$$\leq \frac{1}{2}\sqrt{\int_0^1 \log^2(x)f^2(x)dx}\sqrt{\int_0^1 \log^2(x)g^2(x)dx}.$$

Proof. Several techniques can be used for this proof (including a direct Cauchy-Schwarz inequality for double integrals). Here, to highlight some interesting facts, we will use probability tools, including the notions of bivariate and univariate probability density functions, and the Cauchy-Schwarz inequality in the setting of pairs of random variables.

We set

$$h(x,y) := \frac{1}{x+y}\log\left(\frac{1}{x+y}\right), \quad (x,y) \in \mathcal{S},$$

where $\mathcal{S} = \{(x,y) \in (0,+\infty)^2;\ x+y \leq 1\}$, and $h(x,y) = 0$ for any $(x,y) \notin \mathcal{S}$. Let us show that $h(x,y)$ is a valid bivariate probability density function.

First, for any $(x,y) \in \mathcal{S}$, we have $\log[1/(x+y)] = -\log(x+y) \geq 0$, implying that $h(x,y) \geq 0$. Since $h(x,y) = 0$ for any $(x,y) \notin \mathcal{S}$, we have $h(x,y) \geq 0$ for any $(x,y) \in \mathbb{R}^2$.

Second, with the use of appropriate primitives, the following integral developments hold:

$$\begin{aligned}
&\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} h(x,y)dxdy = \int_0^1\int_0^1 h(x,y)dxdy \\
&= \int_0^1 \left[\int_0^{1-y} \frac{1}{x+y}\log\left(\frac{1}{x+y}\right)dx\right]dy \\
&= \int_0^1 \left[-\frac{1}{2}\log^2(x+y)\right]_{x=0}^{x=1-y} dy \\
&= \frac{1}{2}\int_0^1 \log^2(y)dy = \frac{1}{2}\left[2y + y\log^2(y) - 2y\log(y)\right]_{y=0}^{y=1} = \frac{1}{2}\times(2+0) = 1.
\end{aligned}$$

Since 1 is reached, we conclude that $h(x,y)$ is a valid bivariate probability density function. To the best of our knowledge, it has not been repertorized in the literature, so this is an independent result of interest.

We now introduce two random variables, denoted by X and Y, so that (X,Y) has the probability density function $h(x,y)$. Given the mathematical expectation operator denoted by $\mathbb{E}$ and the fact that f and g are positive, the Cauchy-Schwarz inequality applied to (X,Y) gives

$$\mathbb{E}[f(X)g(Y)] = |\mathbb{E}[f(X)g(Y)]| \leq \sqrt{\mathbb{E}[f^2(X)]}\sqrt{\mathbb{E}[g^2(Y)]}.$$

This corresponds to the following integral inequality:

$$\begin{aligned}
&\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(x)g(y)h(x,y)dxdy \\
&\leq \sqrt{\int_{-\infty}^{+\infty} f^2(x)\phi(x)dx}\sqrt{\int_{-\infty}^{+\infty} g^2(y)\varphi(y)dy}, \qquad (2.1)
\end{aligned}$$

where ϕ denotes the probability density function of X, i.e., $\phi(x) := \int_{-\infty}^{+\infty} h(x,t)dt$ for $x \in \mathbb{R}$, and φ denotes the probability density function of Y, i.e., $\varphi(y) := \int_{-\infty}^{+\infty} h(t,y)dt$ for $y \in \mathbb{R}$. Let us determine the expressions of these two functions.

- For any $x \in (0,1)$, by taking into account the definition of $\mathcal{S}$, we have

$$\phi(x) = \int_{-\infty}^{+\infty} h(x,t)dt = \int_0^{1-x} \frac{1}{x+t} \log\left(\frac{1}{x+t}\right) dt$$
$$= \left[-\frac{1}{2}\log^2(x+t)\right]_{t=0}^{t=1-x} = \frac{1}{2}\log^2(x),$$

and $\phi(x) = 0$ for any $x \notin (0,1)$,

- Proceeding in a similar manner, for any $y \in (0,1)$, we have

$$\varphi(y) = \int_{-\infty}^{+\infty} h(t,y)dt = \int_0^{1-y} \frac{1}{y+t} \log\left(\frac{1}{y+t}\right) dt$$
$$= \phi(y) = \frac{1}{2}\log^2(y),$$

and $\varphi(y) = 0$ for any $y \notin (0,1)$.

Considering the inequality in Equation (2.1) with these expressions, we find that

$$\int\int_{\{(x,y)\in(0,+\infty)^2;\ x+y\le 1\}} \frac{1}{x+y} \log\left(\frac{1}{x+y}\right) f(x)g(y)dxdy$$
$$= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(x)g(y)h(x,y)dxdy$$
$$\le \sqrt{\int_{-\infty}^{+\infty} f^2(x)\phi(x)dx}\sqrt{\int_{-\infty}^{+\infty} g^2(y)\varphi(y)dy}$$
$$= \frac{1}{2}\sqrt{\int_0^1 \log^2(x)f^2(x)dx}\sqrt{\int_0^1 \log^2(y)g^2(y)dy}.$$

Standardizing the notations x and y for the upper bound, we conclude the proof of Theorem 2.1. ∎

We claim that the upper bound in this theorem is sharp. As a sketch, if we take f and g equal to the same constant value, then the lower and upper bounds are equal, i.e., by denoting c the common constant, we find $c^2 \le c^2$. We now strengthen the precision claim by considering two numerical examples.

Example 1: For $f(x) = e^x$ and $g(y) = e^y$, we have the following numerical approximation:

$$\int\int_{\{(x,y)\in(0,+\infty)^2;\ x+y\le 1\}} \frac{1}{x+y} \log\left(\frac{1}{x+y}\right) f(x)g(y)dxdy$$
$$= \int\int_{\{(x,y)\in(0,+\infty)^2;\ x+y\le 1\}} \frac{1}{x+y} \log\left(\frac{1}{x+y}\right) e^{x+y}dxdy$$
$$\approx 1.3179$$

and

$$\int_0^1 \log^2(x)f^2(x)dx = \int_0^1 \log^2(x)g^2(x)dx = \int_0^1 \log^2(x)e^{-2x}dx \approx 2.70359.$$

We thus have $1.3179 \le (1/2)\sqrt{2.70359}\sqrt{2.70359} \approx 1.351795$, which is sharp.

Example 2: For $f(x) = e^{-x}$ and $g(y) = e^{-y}$, we have

$$\int\int_{\{(x,y)\in(0,+\infty)^2;\ x+y\leq 1\}} \frac{1}{x+y}\log\left(\frac{1}{x+y}\right) f(x)g(y)dxdy$$
$$= \int\int_{\{(x,y)\in(0,+\infty)^2;\ x+y\leq 1\}} \frac{1}{x+y}\log\left(\frac{1}{x+y}\right) e^{-x-y}dxdy$$
$$\approx 0.7966$$

and

$$\int_0^1 \log^2(x) f^2(x)dx = \int_0^1 \log^2(x) g^2(x)dx = \int_0^1 \log^2(x)e^{-2x}dx \approx 1.61511.$$

We thus have $0.7966 \leq (1/2)\sqrt{1.61511}\sqrt{1.61511} \approx 0.807555$, which is sharp too.

Several integral techniques have been tested, such as those presented in [14], but the results are not as sharp as those in Theorem 2.1. This is discussed in more detail in Remark 3.5.

We end this part with a remark concerning a possible norm generalization of Theorem 2.1.

Remark 2.2. In the proof of Theorem 2.1, by using the general Hölder inequality for a pair of random variables, instead of the Cauchy-Schwarz inequality (still for a pair of random variables), for any $p, q > 1$ such that $1/p + 1/q = 1$, we get

$$\int\int_{\{(x,y)\in(0,+\infty)^2;\ x+y\leq 1\}} \frac{1}{x+y}\log\left(\frac{1}{x+y}\right) f(x)g(y)dxdy$$
$$\leq \frac{1}{2}\left[\int_0^1 \log^2(x) f^p(x)dx\right]^{1/p}\left[\int_0^1 \log^2(x) g^q(x)dx\right]^{1/q}.$$

Theorem 2.1 follows by taking $p = q = 2$. Since we focus only on the weighted L_2 norms of f and g, we mention this possible generalization as a secondary remark.

2.2. Derived propositions

The proposition below gives a simple lower bound for a specific logarithmic integral inequality.

Proposition 2.3. *Let $f, g : (0, +\infty) \to (0, +\infty)$ be two functions. Assuming that the integrals involved converge, we have*

$$\int_0^{+\infty}\int_0^{+\infty} \frac{1}{x+y}\log\left(\frac{1}{x+y}\right) f(x)g(y)dxdy$$
$$\leq \frac{1}{2}\sqrt{\int_0^1 \log^2(x) f^2(x)dx}\sqrt{\int_0^1 \log^2(x) g^2(x)dx}.$$

Proof. For any $x, y > 0$ such that $x + y > 1$, we have $\log[1/(x + y)] < 0$. This, with the fact that f and g are positive, and Theorem 2.1, implies that

$$\begin{aligned}
&\int_0^{+\infty}\int_0^{+\infty} \frac{1}{x+y}\log\left(\frac{1}{x+y}\right) f(x)g(y)dxdy \\
&= \int\int_{\{(x,y)\in(0,+\infty)^2;\ x+y\leq 1\}} \frac{1}{x+y}\log\left(\frac{1}{x+y}\right) f(x)g(y)dxdy \\
&+ \int\int_{\{(x,y)\in(0,+\infty)^2;\ x+y> 1\}} \frac{1}{x+y}\log\left(\frac{1}{x+y}\right) f(x)g(y)dxdy \\
&\leq \int\int_{\{(x,y)\in(0,+\infty)^2;\ x+y\leq 1\}} \frac{1}{x+y}\log\left(\frac{1}{x+y}\right) f(x)g(y)dxdy \\
&\leq \frac{1}{2}\sqrt{\int_0^1 \log^2(x)f^2(x)dx}\sqrt{\int_0^1 \log^2(x)g^2(x)dx}.
\end{aligned}$$

This concludes the proof. ∎

We now propose an elegant logarithmic integral inequality derived immediately from Theorem 2.1 that involves the exact L_2 norms of f and g.

Proposition 2.4. *Let $f, g : (0, 1) \to (0, +\infty)$ be two functions. Assuming that the integrals involved converge, we have*

$$\begin{aligned}
&\int\int_{\{(x,y)\in(0,+\infty)^2;\ x+y\leq 1\}} \frac{1}{x+y}\log\left(\frac{1}{x+y}\right) \frac{1}{\log(x)\log(y)} f(x)g(y)dxdy \\
&\leq \frac{1}{2}\sqrt{\int_0^1 f^2(x)dx}\sqrt{\int_0^1 g^2(x)dx}.
\end{aligned}$$

Proof. Theorem 2.1 applied with the positive functions $f_\triangle(x) = -f(x)/\log(x)$ and $g_\triangle(y) = -g(y)/\log(y)$ yields

$$\begin{aligned}
&\int\int_{\{(x,y)\in(0,+\infty)^2;\ x+y\leq 1\}} \frac{1}{x+y}\log\left(\frac{1}{x+y}\right) \frac{1}{\log(x)\log(y)} f(x)g(y)dxdy \\
&= \int\int_{\{(x,y)\in(0,+\infty)^2;\ x+y\leq 1\}} \frac{1}{x+y}\log\left(\frac{1}{x+y}\right) f_\triangle(x)g_\triangle(y)dxdy \\
&\leq \frac{1}{2}\sqrt{\int_0^1 \log^2(x)f_\triangle^2(x)dx}\sqrt{\int_0^1 \log^2(x)g_\triangle^2(x)dx} \\
&= \frac{1}{2}\sqrt{\int_0^1 f^2(x)dx}\sqrt{\int_0^1 g^2(x)dx}.
\end{aligned}$$

This concludes the proof of Proposition 2.4. ∎

The proposition below gives a simple lower bound for a specific logarithmic integral inequality.

Proposition 2.5. *Let $f, g : (0, 1) \to (0, +\infty)$ be two functions. Assuming that the integrals involved converge, we have*

$$\int\int_{\{(x,y)\in(0,+\infty)^2;\ x+y\leq 1\}} \frac{1}{x+y} \log(x+y) f(x)g(y)dxdy$$
$$\geq -\frac{1}{2}\sqrt{\int_0^1 \log^2(x) f^2(x)dx}\sqrt{\int_0^1 \log^2(x) g^2(x)dx}.$$

Proof. The proof follows immediately from Theorem 2.1 and the fact that $\log[1/(x+y)] = -\log(x+y)$. ■

Through mathematical analysis, adjustable parameters can be introduced into the result of Theorem 2.1. The proposition below illustrates this claim.

Proposition 2.6. *Let $\alpha, \beta > 0$ and $f, g : (0, +\infty) \to (0, +\infty)$ be two functions. Assuming that the integrals involved converge, we have*

$$\int\int_{\{(x,y)\in(0,+\infty)^2;\ x+y\leq \alpha\}} \frac{1}{x+y} \log\left[\frac{\alpha^\beta}{(x+y)^\beta}\right] f(x)g(y)dxdy$$
$$\leq \frac{\beta}{2}\sqrt{\int_0^\alpha \log^2\left(\frac{x}{\alpha}\right) f^2(x)dx}\sqrt{\int_0^\alpha \log^2\left(\frac{x}{\alpha}\right) g^2(x)dx}.$$

Proof. First, by using a basic property of the logarithm function, we note that

$$\int\int_{\{(x,y)\in(0,+\infty)^2;\ x+y\leq \alpha\}} \frac{1}{x+y} \log\left[\frac{\alpha^\beta}{(x+y)^\beta}\right] f(x)g(y)dxdy$$
$$= \int\int_{\{(x,y)\in(0,+\infty)^2;\ x+y\leq \alpha\}} \frac{1}{x+y} \log\left[\left(\frac{\alpha}{x+y}\right)^\beta\right] f(x)g(y)dxdy$$
$$= \beta \int\int_{\{(x,y)\in(0,+\infty)^2;\ x+y\leq \alpha\}} \frac{1}{x+y} \log\left(\frac{\alpha}{x+y}\right) f(x)g(y)dxdy. \tag{2.2}$$

Then, using the change of variables $(x, y) = (\alpha u, \alpha v)$, we get

$$\int\int_{\{(u,v)\in(0,+\infty)^2;\ \alpha u+\alpha v\leq \alpha\}} \frac{1}{\alpha(u+v)} \log\left[\frac{\alpha}{\alpha(u+v)}\right] f(\alpha u)g(\alpha v)\alpha^2 dudv$$
$$= \alpha \int\int_{\{(u,v)\in(0,+\infty)^2;\ u+v\leq 1\}} \frac{1}{u+v} \log\left(\frac{1}{u+v}\right) f_\star(u)g_\star(v)dudv, \tag{2.3}$$

where $f_\star(u) = f(\alpha u)$ and $g_\star(v) = g(\alpha v)$. It follows from Theorem 2.1 applied with $f_\star$ and $g_\star$, and the "backward" change of variables $u = w/\alpha$ and $v = z/\alpha$ that

$$\begin{aligned}
&\int\int_{\{(u,v)\in(0,+\infty)^2;\ u+v\le 1\}} \frac{1}{u+v}\log\left(\frac{1}{u+v}\right) f_\star(u)g_\star(v)dudv \\
&\le \frac{1}{2}\sqrt{\int_0^1 \log^2(u)f_\star^2(u)du}\sqrt{\int_0^1 \log^2(v)g_\star^2(v)dv} \\
&= \frac{1}{2}\sqrt{\int_0^\alpha \log^2\left(\frac{w}{\alpha}\right) f_\star^2\left(\frac{w}{\alpha}\right)\frac{1}{\alpha}dw}\sqrt{\int_0^\alpha \log^2\left(\frac{z}{\alpha}\right) g_\star^2\left(\frac{z}{\alpha}\right)\frac{1}{\alpha}dz} \\
&= \frac{1}{2\alpha}\sqrt{\int_0^\alpha \log^2\left(\frac{x}{\alpha}\right) f^2(x)dx}\sqrt{\int_0^\alpha \log^2\left(\frac{x}{\alpha}\right) g^2(x)dx}. \qquad (2.4)
\end{aligned}$$

Putting Equations (2.2), (2.3) and (2.4) together, we end the proof of Proposition 2.6. ■

In particular, for any positive integer m, applying the binomial theorem and Proposition 2.6 with $\alpha = 1$ and $\beta = m$, we have

$$\begin{aligned}
&\int\int_{\{(x,y)\in(0,+\infty)^2;\ x+y\le 1\}} \frac{1}{x+y}\log\left[\frac{1}{\sum_{k=0}^m \binom{m}{k} x^k y^{m-k}}\right] f(x)g(y)dxdy \\
&= \int\int_{\{(x,y)\in(0,+\infty)^2;\ x+y\le 1\}} \frac{1}{x+y}\log\left[\frac{1}{(x+y)^m}\right] f(x)g(y)dxdy \\
&\le \frac{m}{2}\sqrt{\int_0^1 \log^2(x)f^2(x)dx}\sqrt{\int_0^1 \log^2(x)g^2(x)dx},
\end{aligned}$$

where $\binom{m}{k}$ is the binomial coefficient with parameters k and m, i.e., $\binom{m}{k} = m!/[k!(m-k)!]$.

The proposition below offers another way of looking at Theorem 2.1, with a kind of integration domain counterpart.

Proposition 2.7. *Let $f, g : (1, +\infty) \to (0, +\infty)$ be two functions. Assuming that the integrals involved converge, we have*

$$\begin{aligned}
&\int\int_{\{(x,y)\in(0,+\infty)^2;\ x+y\le xy\}} \frac{1}{xy(x+y)}\log\left(\frac{xy}{x+y}\right) f(x)g(y)dxdy \\
&\le \frac{1}{2}\sqrt{\int_1^{+\infty} \frac{1}{x^2}\log^2(x)f^2(x)dx}\sqrt{\int_1^{+\infty} \frac{1}{x^2}\log^2(x)g^2(x)dx}.
\end{aligned}$$

Proof. First we notice that $\{(x,y)\in(0,+\infty)^2;\ x+y\leq xy\}=\{(x,y)\in(0,+\infty)^2;\ 1/x+1/y\leq 1\}$. Using the change of variables $(x,y)=(1/u,1/v)$, we get

$$\begin{aligned}
&\int\int_{\{(x,y)\in(0,+\infty)^2;\ x+y\leq xy\}}\frac{1}{xy(x+y)}\log\left(\frac{xy}{x+y}\right)f(x)g(y)dxdy\\
&=\int\int_{\{(x,y)\in(0,+\infty)^2;\ 1/x+1/y\leq 1\}}\frac{1}{1/x+1/y}\log\left(\frac{1}{1/x+1/y}\right)\\
&\times f(x)g(y)\left(-\frac{1}{x^2}\right)\left(-\frac{1}{y^2}\right)dxdy\\
&=\int\int_{\{(u,v)\in(0,+\infty)^2;\ u+v\leq 1\}}\frac{1}{u+v}\log\left(\frac{1}{u+v}\right)f_\dagger(u)g_\dagger(v)dudv, \qquad (2.5)
\end{aligned}$$

where $f_\dagger(u)=f(1/u)$ and $g_\dagger(v)=g(1/v)$. It follows from Theorem 2.1 applied with $f_\dagger$ and $g_\dagger$, and the "backward" change of variables $u=1/w$ and $v=1/z$ that

$$\begin{aligned}
&\int\int_{\{(u,v)\in(0,+\infty)^2;\ u+v\leq 1\}}\frac{1}{u+v}\log\left(\frac{1}{u+v}\right)f_\dagger(u)g_\dagger(v)dudv\\
&\leq\frac{1}{2}\sqrt{\int_0^1\log^2(u)f_\dagger^2(u)du}\sqrt{\int_0^1\log^2(v)g_\dagger^2(v)dv}\\
&=\frac{1}{2}\sqrt{\int_1^{+\infty}\log^2(w)f_\dagger^2\left(\frac{1}{w}\right)\frac{1}{w^2}dw}\sqrt{\int_1^{+\infty}\log^2(z)g_\dagger^2\left(\frac{1}{z}\right)\frac{1}{z^2}dz}\\
&=\frac{1}{2}\sqrt{\int_1^{+\infty}\frac{1}{x^2}\log^2(x)f^2(x)dx}\sqrt{\int_1^{+\infty}\frac{1}{x^2}\log^2(x)g^2(x)dx}. \qquad (2.6)
\end{aligned}$$

Putting Equations (2.5) and (2.6) concludes the proof. ■

The second main theorem completes the first one in a way. It is the subject of the next section.

3. Second theorem and derived results

3.1. Second logarithmic integral inequality

The theorem below considers a new logarithmic integral inequality which can be viewed as the counterpart of Theorem 2.1 (this aspect will be discussed later). As mentioned earlier, the Lobachevskii function appears quite naturally in the upper bound obtained.

Theorem 3.1. *Let $f,g:(1,+\infty)\to(0,+\infty)$ be two functions. Assuming that the integrals involved converge, we have*

$$\begin{aligned}
&\int\int_{\{(x,y)\in(0,+\infty)^2;\ x+y>1\}}\frac{1}{x+y}\log(x+y)f(x)g(y)dxdy\\
&\leq 4\sqrt{\int_0^{+\infty}\psi(x)f^2(x)dx}\sqrt{\int_0^{+\infty}\psi(x)g^2(x)dx},
\end{aligned}$$

where

$$\psi(x) := \left\{\frac{1}{2}\log(x)\arcsin[\sqrt{x}] - \mathcal{L}\left\{\arccos[\sqrt{x}]\right\}\right\} \mathbf{1}_{(0,1)}(x) \\ + \frac{\pi}{4}\log(w)\mathbf{1}_{(1,+\infty)}(x) + \frac{\pi}{2}\log(2)\mathbf{1}_{(0,+\infty)}(x), \tag{3.1}$$

$\mathbf{1}_{\mathcal{T}}(w)$ denotes the indicator function on a certain set $\mathcal{T}$, i.e., $\mathbf{1}_{\mathcal{T}}(w) = 1$ if $w \in \mathcal{T}$, and $\mathbf{1}_{\mathcal{T}}(w) = 0$ if $w \notin \mathcal{T}$, and

$$\mathcal{L}(x) := -\int_0^x \log[\cos(t)]dt, \tag{3.2}$$

with $x \in [-\pi/2, \pi/2]$, is the Lobachevskii function defined in [8, Item 184(10)] (see also in [5, Item 8.260], where diverse representations and functional relationships of this function are given, as well as in the appendix of this paper).

Proof. Let us consider the following double integral for brevity:

$$I := \int_{\{(x,y)\in(0,+\infty)^2;\ x+y>1\}} \frac{1}{x+y}\log(x+y)f(x)g(y)dxdy. \tag{3.3}$$

Using the change of variables $(x, y) = (p^2, q^2)$, we obtain

$$I = 4\int\int_{\{(p,q)\in(0,+\infty)^2;\ p^2+q^2>1\}} \frac{pq}{p^2+q^2}\log\left[p^2+q^2\right] f(p^2)g(q^2)dpdq.$$

Based on this expression, applying the changes of variables $(p, q) = (r\cos(\theta), r\sin(\theta))$, with $\{(p, q) \in (0, +\infty)^2;\ p^2 + q^2 > 1\} = \{(r, \theta) \in (0, +\infty)^2;\ r > 1,\ \theta \in [0, \pi/2]\}$, and noticing that $\log(r) > 0$ for any $r > 1$, we obtain

$$\begin{aligned} I &= 4\int_0^{\pi/2}\int_1^{+\infty} \frac{r\cos(\theta)r\sin(\theta)}{r^2\cos^2(\theta)+r^2\sin^2(\theta)}\log\left[r^2\cos^2(\theta)+r^2\sin^2(\theta)\right] \times \\ &\quad f[r^2\cos^2(\theta)]g[r^2\cos^2(\theta)]rdrd\theta \\ &= 8\int_0^{\pi/2}\int_1^{+\infty} \cos(\theta)\sin(\theta)r\log(r)f[r^2\cos^2(\theta)]g[r^2\sin^2(\theta)]drd\theta. \end{aligned}$$

Applying the Fubini-Tonelli theorem, the Cauchy-Schwarz inequality only with respect to r, and distinguishing the variable r in each of the integrals obtained, we have

$$\begin{aligned} I &= 8\int_0^{\pi/2}\cos(\theta)\sin(\theta)\left\{\int_1^{+\infty} r\log(r)f[r^2\cos^2(\theta)]g[r^2\sin^2(\theta)]dr\right\}d\theta \\ &\leq 8\int_0^{\pi/2}\cos(\theta)\sin(\theta)\sqrt{\int_1^{+\infty} r\log(r)f^2[r^2\cos^2(\theta)]dr}\times \\ &\quad \sqrt{\int_1^{+\infty} r\log(r)g^2[r^2\sin^2(\theta)]drd\theta} \\ &= 8J, \end{aligned} \tag{3.4}$$

where

$$J := \int_0^{\pi/2} \cos(\theta)\sin(\theta)\sqrt{\int_1^{+\infty} r_1 \log(r_1) f^2[r_1^2\cos^2(\theta)]dr_1} \times \sqrt{\int_1^{+\infty} r_2 \log(r_2) g^2[r_2^2\sin^2(\theta)]dr_2}\, d\theta.$$

Now let us re-express J in a way that isolates f^2 and g^2. By the change of variables $w = r_1^2\cos^2(\theta)$, we get

$$\int_1^{+\infty} r_1 \log(r_1) f^2[r_1^2\cos^2(\theta)]dr_1 = \frac{1}{2\cos^2(\theta)}\int_{\cos^2(\theta)}^{+\infty}\left\{\frac{1}{2}\log(w) - \log[\cos(\theta)]\right\} f^2(w)dw.$$

Similarly, applying the change of variables $z = r_2^2\sin^2(\theta)$, we have

$$\int_1^{+\infty} r_2 \log(r_2) g^2[r_2^2\sin^2(\theta)]dr_2 = \frac{1}{2\sin^2(\theta)}\int_{\sin^2(\theta)}^{+\infty}\left\{\frac{1}{2}\log(z) - \log[\sin(\theta)]\right\} g^2(z)dz.$$

We thus have

$$J = \frac{1}{2}\int_0^{\pi/2} \sqrt{\int_{\cos^2(\theta)}^{+\infty}\left\{\frac{1}{2}\log(w) - \log[\cos(\theta)]\right\} f^2(w)dw} \times \sqrt{\int_{\sin^2(\theta)}^{+\infty}\left\{\frac{1}{2}\log(z) - \log[\sin(\theta)]\right\} g^2(z)dz}\, d\theta.$$

Based on this expression, we want to find a sharp upper bound for J. Using the Cauchy-Schwarz inequality with respect to θ, we get

$$J \leq \frac{1}{2}\sqrt{K}\sqrt{L}, \tag{3.5}$$

where

$$K := \int_0^{\pi/2}\int_{\cos^2(\theta)}^{+\infty}\left\{\frac{1}{2}\log(w) - \log[\cos(\theta)]\right\} f^2(w)dwd\theta$$

and

$$L := \int_0^{\pi/2}\int_{\sin^2(\theta)}^{+\infty}\left\{\frac{1}{2}\log(z) - \log[\sin(\theta)]\right\} g^2(z)dzd\theta.$$

Let us investigate these terms, successively.

For the term K, by the Chasles integral relation, we can write

$$K = M + N, \tag{3.6}$$

where

$$M := \int_0^{\pi/2}\int_{\cos^2(\theta)}^{1}\left\{\frac{1}{2}\log(w) - \log[\cos(\theta)]\right\} f^2(w)dwd\theta$$

and

$$N := \int_0^{\pi/2} \int_1^{+\infty} \left\{ \frac{1}{2} \log(w) - \log[\cos(\theta)] \right\} f^2(w) dw d\theta.$$

The Lobachevskii function $\mathcal{L}$, as introduced in Equation (3.2), is crucial for dealing with the term M. Exchanging the order of integration thanks to the Fubini-Tonelli theorem, and using $\pi/2 - \arccos(x) = \arcsin(x)$ and $\mathcal{L}(\pi/2) = (\pi/2)\log(2)$, we get

$$\begin{aligned} M &= \int_0^1 \left[\int_{\arccos[\sqrt{w}]}^{\pi/2} \left\{ \frac{1}{2} \log(w) - \log[\cos(\theta)] \right\} d\theta \right] f^2(w) dw \\ &= \int_0^1 \left[\frac{1}{2} \log(w) \left\{ \int_{\arccos[\sqrt{w}]}^{\pi/2} d\theta \right\} - \int_{\arccos[\sqrt{w}]}^{\pi/2} \log[\cos(\theta)] d\theta \right] f^2(w) dw \\ &= \int_0^1 \left\{ \frac{1}{2} \log(w) \arcsin[\sqrt{w}] + \mathcal{L}\left(\frac{\pi}{2}\right) - \mathcal{L}\left\{\arccos[\sqrt{w}]\right\} \right\} f^2(w) dw \\ &= \int_0^1 \left\{ \frac{1}{2} \log(w) \arcsin[\sqrt{w}] + \frac{\pi}{2} \log(2) - \mathcal{L}\left\{\arccos[\sqrt{w}]\right\} \right\} f^2(w) dw. \end{aligned}$$

For the term N, the development is more direct. We have

$$\begin{aligned} N &= \int_1^{+\infty} \left[\int_0^{\pi/2} \left\{ \frac{1}{2} \log(w) - \log[\cos(\theta)] \right\} d\theta \right] f^2(w) dw \\ &= \int_1^{+\infty} \left[\frac{\pi}{4} \log(w) + \mathcal{L}\left(\frac{\pi}{2}\right) \right] f^2(w) dw \\ &= \int_1^{+\infty} \left[\frac{\pi}{4} \log(w) + \frac{\pi}{2} \log(2) \right] f^2(w) dw. \end{aligned}$$

Hence, by merging the expressions obtained for M and N, we find that

$$\begin{aligned} K &= \int_0^1 \left\{ \frac{1}{2} \log(w) \arcsin[\sqrt{w}] - \mathcal{L}\left\{\arccos[\sqrt{w}]\right\} \right\} f^2(w) dw \\ &\quad + \frac{\pi}{4} \int_1^{+\infty} \log(w) f^2(w) dw + \frac{\pi}{2} \log(2) \int_0^{+\infty} f^2(w) dw \\ &= \int_0^{+\infty} \psi(w) f^2(w) dw, \end{aligned} \tag{3.7}$$

where ψ is defined in Equation (3.1).

For the term L, since $\sin(\pi/2 - \theta) = \cos(\theta)$, the change of variables $\theta = \pi/2 - \tau$ gives

$$L = \int_0^{\pi/2} \int_{\cos^2(\tau)}^{+\infty} \left\{ \frac{1}{2} \log(z) - \log[\cos(\tau)] \right\} g^2(z) dz d\tau.$$

We recognize the expression of the term K with g instead of f. Based on this remark and Equation (3.7), we get

$$L = \int_0^{+\infty} \psi(z) g^2(z) dz, \tag{3.8}$$

where ψ is defined in Equation (3.1).

Combining Equations (3.3), (3.4), (3.5), (3.7) and (3.8) and uniformizing the notations, we obtain

$$\int\int_{\{(x,y)\in(0,+\infty)^2;\ x+y>1\}} \frac{1}{x+y}\log(x+y)f(x)g(y)dxdy$$
$$\leq 4\sqrt{\int_0^{+\infty}\psi(x)f^2(x)dx}\sqrt{\int_0^{+\infty}\psi(x)g^2(x)dx}.$$

The proof of Theorem 3.1 ends. ■

We thus have an upper bound depending on the weighted L_2 norms of f and g, with a weight depending on the Lobachevskii function.

Some derived integral inequalities are presented in the next section.

3.2. Derived propositions

The proposition below gives a simple lower bound for a certain logarithmic integral inequality.

Proposition 3.2. *Let $f, g : (1, +\infty) \to (0, +\infty)$ be two functions. Assuming that the integrals involved converge, we have*

$$\int\int_{\{(x,y)\in(0,+\infty)^2;\ x+y>1\}} \frac{1}{x+y}\log\left(\frac{1}{x+y}\right)f(x)g(y)dxdy$$
$$\geq -4\sqrt{\int_0^{+\infty}\psi(x)f^2(x)dx}\sqrt{\int_0^{+\infty}\psi(x)g^2(x)dx},$$

where ψ is given in Equation (3.1).

Proof. The proof follows directly from Theorem 3.1 and the fact that $\log[1/(x+y)] = -\log(x+y)$. ■

In the spirit of Proposition 2.6, with some mathematical effort, adjustable parameters can be introduced into the result of Theorem 3.1, as shown in the proposition below.

Proposition 3.3. *Let $\alpha, \beta > 0$ and $f, g : (0, +\infty) \to (0, +\infty)$ be two functions. Assuming that the integrals involved converge, we have*

$$\int\int_{\{(x,y)\in(0,+\infty)^2;\ x+y>\alpha\}} \frac{1}{x+y}\log\left[\frac{(x+y)^\beta}{\alpha^\beta}\right]f(x)g(y)dxdy$$
$$\leq 4\beta\sqrt{\int_0^{+\infty}\psi\left(\frac{x}{\alpha}\right)f^2(x)dx}\sqrt{\int_0^{+\infty}\psi\left(\frac{x}{\alpha}\right)g^2(x)dx},$$

where ψ is given in Equation (3.1).

Proof. By means of a basic logarithmic property, we have

$$\int\int_{\{(x,y)\in(0,+\infty)^2;\ x+y>\alpha\}} \frac{1}{x+y}\log\left[\frac{(x+y)^\beta}{\alpha^\beta}\right]f(x)g(y)dxdy$$
$$= \beta\int\int_{\{(x,y)\in(0,+\infty)^2;\ x+y>\alpha\}} \frac{1}{x+y}\log\left(\frac{x+y}{\alpha}\right)f(x)g(y)dxdy. \qquad (3.9)$$

Then, using the change of variables $(x, y) = (\alpha u, \alpha v)$, we get

$$\begin{aligned}
&\int\int_{\{(u,v)\in(0,+\infty)^2;\ \alpha u+\alpha v>\alpha\}} \frac{1}{\alpha(u+v)} \log\left[\frac{\alpha(u+v)}{\alpha}\right] f(\alpha u)g(\alpha v)\alpha^2 dudv \\
&= \alpha \int\int_{\{(u,v)\in(0,+\infty)^2;\ u+v>1\}} \frac{1}{u+v} \log(u+v) f_\star(u) g_\star(v) dudv, \qquad (3.10)
\end{aligned}$$

where $f_\star(u) = f(\alpha u)$ and $g_\star(v) = g(\alpha v)$. Applying Theorem 3.1 with $f_\star$ and $g_\star$, and the "backward" change of variables $u = w/\alpha$ and $v = z/\alpha$, we obtain

$$\begin{aligned}
&\int\int_{\{(u,v)\in(0,+\infty)^2;\ u+v>1\}} \frac{1}{u+v} \log(u+v) f_\star(u) g_\star(v) dudv \\
&\leq 4\sqrt{\int_0^{+\infty} \psi(u) f_\star^2(u) du}\sqrt{\int_0^{+\infty} \psi(v) g_\star^2(v) dv} \\
&= 4\sqrt{\int_0^{+\infty} \psi\left(\frac{w}{\alpha}\right) f_\star^2\left(\frac{w}{\alpha}\right)\frac{1}{\alpha} dw}\sqrt{\int_0^{+\infty} \psi\left(\frac{z}{\alpha}\right) g_\star^2\left(\frac{z}{\alpha}\right)\frac{1}{\alpha} dz} \\
&= \frac{4}{\alpha}\sqrt{\int_0^{+\infty} \psi\left(\frac{x}{\alpha}\right) f^2(x) dx}\sqrt{\int_0^{+\infty} \psi\left(\frac{x}{\alpha}\right) g^2(x) dx}. \qquad (3.11)
\end{aligned}$$

Combining Equations (3.9), (3.10) and (3.11) together, we conclude the proof of Proposition 3.3. ■

In particular, for any positive integer m, applying the binomial theorem and Proposition 3.3 with $\alpha = 1$ and $\beta = m$, we have

$$\begin{aligned}
&\int\int_{\{(x,y)\in(0,+\infty)^2;\ x+y>1\}} \frac{1}{x+y} \log\left[\sum_{k=0}^{m} \binom{m}{k} x^k y^{m-k}\right] f(x)g(y) dxdy \\
&= \int\int_{\{(x,y)\in(0,+\infty)^2;\ x+y>1\}} \frac{1}{x+y} \log\left[(x+y)^m\right] f(x)g(y) dxdy \\
&\leq 4m\sqrt{\int_0^{+\infty} \psi(x) f^2(x) dx}\sqrt{\int_0^{+\infty} \psi(x) g^2(x) dx}.
\end{aligned}$$

In the same way as Proposition 2.7, the proposition below offers another way of apprehending Theorem 3.1, with a kind of integration domain counterpart.

Proposition 3.4. *Let $f, g : (0, +\infty) \to (0, +\infty)$ be two functions. Assuming that the integrals involved converge, we have*

$$\begin{aligned}
&\int\int_{\{(x,y)\in(0,+\infty)^2;\ x+y>xy\}} \frac{1}{xy(x+y)} \log\left(\frac{x+y}{xy}\right) f(x)g(y) dxdy \\
&\leq 4\sqrt{\int_0^{+\infty} \frac{1}{x^2}\psi\left(\frac{1}{x}\right) f^2(x) dx}\sqrt{\int_0^{+\infty} \frac{1}{x^2}\psi\left(\frac{1}{x}\right) g^2(x) dx},
\end{aligned}$$

where ψ is given in Equation (3.1).

Proof. First, let us remark that $\{(x,y)\in(0,+\infty)^2;\ x+y>xy\}=\{(x,y)\in(0,+\infty)^2;\ 1/x+1/y>1\}$. Using the change of variables $(x,y)=(1/u,1/v)$, we get

$$\begin{aligned}
&\int\int_{\{(x,y)\in(0,+\infty)^2;\ x+y>xy\}}\frac{1}{xy(x+y)}\log\left(\frac{x+y}{xy}\right)f(x)g(y)dxdy\\
&=\int\int_{\{(x,y)\in(0,+\infty)^2;\ 1/x+1/y>1\}}\frac{1}{1/x+1/y}\log\left(\frac{1}{x}+\frac{1}{y}\right)\\
&\quad\times\ f(x)g(y)\left(-\frac{1}{x^2}\right)\left(-\frac{1}{y^2}\right)dxdy\\
&=\int\int_{\{(u,v)\in(0,+\infty)^2;\ u+v>1\}}\frac{1}{u+v}\log(u+v)f_\dagger(u)g_\dagger(v)dudv, \qquad (3.12)
\end{aligned}$$

where $f_\dagger(u)=f(1/u)$ and $g_\dagger(v)=g(1/v)$. Based on Theorem 3.1 with $f_\dagger$ and $g_\dagger$, and the "backward" change of variables $u=1/w$ and $v=1/z$, we get

$$\begin{aligned}
&\int\int_{\{(u,v)\in(0,+\infty)^2;\ u+v>1\}}\frac{1}{u+v}\log(u+v)f_\dagger(u)g_\dagger(v)dudv\\
&\leq 4\sqrt{\int_0^{+\infty}\psi(u)f_\dagger^2(u)du}\sqrt{\int_0^{+\infty}\psi(v)g_\dagger^2(v)dv}\\
&=4\sqrt{\int_0^{+\infty}\psi\left(\frac{1}{w}\right)f_\dagger^2\left(\frac{1}{w}\right)\frac{1}{w^2}dw}\sqrt{\int_0^{+\infty}\psi\left(\frac{1}{z}\right)g_\dagger^2\left(\frac{1}{z}\right)\frac{1}{z^2}dz}\\
&=4\sqrt{\int_0^{+\infty}\frac{1}{x^2}\psi\left(\frac{1}{x}\right)f^2(x)dx}\sqrt{\int_0^{+\infty}\frac{1}{x^2}\psi\left(\frac{1}{x}\right)g^2(x)dx}. \qquad (3.13)
\end{aligned}$$

We conclude the proof of Proposition 3.4 by combining Equations (3.12) and (3.13). ∎

We end this part with a remark on the transposition of the techniques in Theorem 3.1 for the purposes of Theorem 2.1.

Remark 3.5. Applying the techniques in the proof of Theorem 3.1 in the context of the double integral considered in Theorem 2.1, we are able to show that

$$\begin{aligned}
&\int\int_{\{(x,y)\in(0,+\infty)^2;\ x+y\leq 1\}}\frac{1}{x+y}\log\left(\frac{1}{x+y}\right)f(x)g(y)dxdy\\
&\leq 4\sqrt{\int_0^1\rho(x)f^2(x)dx}\sqrt{\int_0^1\rho(x)g^2(x)dx},
\end{aligned}$$

where

$$\rho(x)=-\frac{1}{2}\log(x)\arccos[\sqrt{x}]-\mathcal{L}\left\{\arccos[\sqrt{x}]\right\}$$

and $\mathcal{L}$ is the Lobachevskii function given in Equation (3.2). However, after some numerical tests, we see that the upper bound is not as sharp as the one in Theorem 2.1. For example, considering $f(x) = e^x$ and $g(y) = e^y$, we get

$$\int\int_{\{(x,y)\in(0,+\infty)^2;\ x+y\leq 1\}} \frac{1}{x+y} \log\left(\frac{1}{x+y}\right) f(x)g(y)dxdy$$
$$= \int\int_{\{(x,y)\in(0,+\infty)^2;\ x+y\leq 1\}} \frac{1}{x+y} \log\left(\frac{1}{x+y}\right) e^{x+y}dxdy \approx 1.3179$$

and

$$\int_0^1 \phi(x)f^2(x)dx = -\frac{1}{2}\int_0^1 \log(x)\arccos[\sqrt{x}]e^{2x}dx - \int_0^1 \mathcal{L}\left\{\arccos[\sqrt{x}]\right\}e^{2x}dx$$
$$= 0.872302 - 0.253063 \approx 0.619239.$$

As expected, we have $1.3179 \leq 4\times\sqrt{0.619239}\sqrt{0.619239} \approx 2.476956$, but this upper bound is not as sharp as the value 1.351795 obtained by applying Theorem 2.1. On the other hand, the probability techniques used in Theorem 2.1 cannot be applied to the double integral in Theorem 3.1, justifying the use of a different, more sophisticated technique.

The next section is devoted to a global theorem that can be proved by combining the results in Theorems 2.1 and 3.1.

4. Global theorem and derived results

4.1. Global theorem

Thanks to the two previously established theorems, the result below shows acceptable upper bounds for the double integral in Equation (1.1).

Theorem 4.1. *Let $f, g : (0, +\infty) \to (0, +\infty)$ be two functions. Assuming that the integrals involved converge, the two logarithmic integral inequalities can be established.*

(1) We have

$$\int_0^{+\infty}\int_0^{+\infty} \frac{1}{x+y}\left|\log\left(\frac{1}{x+y}\right)\right| f(x)g(y)dxdy$$
$$\leq \frac{1}{2}\sqrt{\int_0^1 \log^2(x)f^2(x)dx}\sqrt{\int_0^1 \log^2(x)g^2(x)dx}$$
$$+\ 4\sqrt{\int_0^{+\infty} \psi(x)f^2(x)dx}\sqrt{\int_0^{+\infty} \psi(x)g^2(x)dx},$$

where ψ is given in Equation (3.1).

(2) Alternatively, in a more standard form, but with an increase in the level of complexity, we have

$$\int_0^{+\infty}\int_0^{+\infty} \frac{1}{x+y}\left|\log\left(\frac{1}{x+y}\right)\right| f(x)g(y)dxdy$$
$$\leq 4\sqrt{\int_0^{+\infty} \omega(x)f^2(x)dx}\sqrt{\int_0^{+\infty} \omega(x)g^2(x)dx},$$

where

$$\omega(x) = \left\{\log(x)\arcsin[\sqrt{x}] - \frac{\pi}{4}\log(x) - 2\mathcal{L}\left\{\arccos[\sqrt{x}]\right\}\right\}\mathbf{1}_{(0,1)}(x) + \frac{\pi}{4}\log(x)\mathbf{1}_{(1,+\infty)}(x) + \frac{\pi}{2}\log(2)\mathbf{1}_{(0,+\infty)}(x). \tag{4.1}$$

We recall that $\mathcal{L}$ is the Lobachevskii function given in Equation (3.2).

Proof. (1) Thanks to the Chasles integral relation, we can write

$$\int_0^{+\infty}\int_0^{+\infty}\frac{1}{x+y}\left|\log\left(\frac{1}{x+y}\right)\right| f(x)g(y)dxdy = Q + R,$$

where

$$Q := \int\int_{\{(x,y)\in(0,+\infty)^2;\ x+y\leq 1\}}\frac{1}{x+y}\log\left(\frac{1}{x+y}\right)f(x)g(y)dxdy$$

and

$$R := \int\int_{\{(x,y)\in(0,+\infty)^2;\ x+y>1\}}\frac{1}{x+y}\log(x+y)f(x)g(y)dxdy.$$

Applying Theorem 2.1 for the term Q and Theorem 3.1 for the term R, we obtain the desired upper bound directly by summation.

(2) For this item, we can reproduce the proof of Theorem 3.1 "line by line", adapting it to the situation. We omit the details for the sake of redundancy. With appropriate changes of variables, the Fubini-Tonelli theorem and the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned} I_\star &:= \int_0^{+\infty}\int_0^{+\infty}\frac{1}{x+y}\left|\log\left(\frac{1}{x+y}\right)\right| f(x)g(y)dxdy \\ &= 8\int_0^{\pi/2}\int_0^{+\infty}\cos(\theta)\sin(\theta)r|\log(r)|f[r^2\cos^2(\theta)]g[r^2\sin^2(\theta)]drd\theta \\ &\leq 8J_\star, \end{aligned} \tag{4.2}$$

where

$$\begin{aligned} J_\star &:= \int_0^{\pi/2}\cos(\theta)\sin(\theta)\sqrt{\int_0^{+\infty} r_1|\log(r_1)|f^2[r_1^2\cos^2(\theta)]dr_1}\times \\ &\quad \sqrt{\int_0^{+\infty} r_2|\log(r_2)|g^2[r_2^2\sin^2(\theta)]dr_2}d\theta \\ &= \frac{1}{2}\int_0^{\pi/2}\sqrt{\int_0^{+\infty}\left|\frac{1}{2}\log(w) - \log[\cos(\theta)]\right| f^2(w)dw}\times \\ &\quad \sqrt{\int_0^{+\infty}\left|\frac{1}{2}\log(z) - \log[\sin(\theta)]\right| g^2(z)dz}d\theta. \end{aligned}$$

By the Cauchy-Schwarz inequality, we have

$$J_\star \leq \frac{1}{2}\sqrt{K_\star}\sqrt{L_\star}, \tag{4.3}$$

where

$$K_\star := \int_0^{\pi/2} \int_0^{+\infty} \left| \frac{1}{2} \log(w) - \log[\cos(\theta)] \right| f^2(w) dw d\theta$$

and

$$L_\star := \int_0^{\pi/2} \int_0^{+\infty} \left| \frac{1}{2} \log(z) - \log[\sin(\theta)] \right| g^2(z) dz d\theta.$$

For the term $K_\star$, having rearranged the integrated variables, we can write

$$K_\star = U + K,$$

where

$$U := \int_0^{\pi/2} \int_0^{\cos^2(\theta)} \left\{ \log[\cos(\theta)] - \frac{1}{2} \log(w) \right\} f^2(w) dw d\theta$$

and K is given in Equation (3.6), and develops in its final form in Equation (3.7).

For the term U, the following decomposition holds:

$$\begin{aligned} U &= \int_0^1 \left[\int_0^{\arccos[\sqrt{w}]} \left\{ \log[\cos(\theta)] - \frac{1}{2} \log(w) \right\} d\theta \right] f^2(w) dw \\ &= \int_0^1 \left[\int_0^{\arccos[\sqrt{w}]} \log[\cos(\theta)] d\theta - \frac{1}{2} \log(w) \left\{ \int_0^{\arccos[\sqrt{w}]} d\theta \right\} \right] f^2(w) dw \\ &= \int_0^1 \left[-\mathcal{L}\left\{ \arccos[\sqrt{w}] \right\} - \frac{1}{2} \log(w) \arccos[\sqrt{w}] \right] f^2(w) dw. \end{aligned}$$

Combining the expressions of U and K, we obtain

$$\begin{aligned} K_\star &= \int_0^{+\infty} \left[-\mathcal{L}\left\{ \arccos[\sqrt{w}] \right\} - \frac{1}{2} \log(w) \arccos[\sqrt{w}] + \psi(w) \right] f^2(w) dw \\ &= \int_0^{+\infty} \omega(w) f^2(w) dw, \end{aligned} \tag{4.4}$$

where ω is defined in Equation (4.1). Note that we used $\arcsin(x) - \arccos(x) = 2\arcsin(x) - \pi/2$.

For the term $L_\star$, since $\sin(\pi/2 - \theta) = \cos(\theta)$, the change of variables $\theta = \pi/2 - \tau$ gives

$$L_\star = \int_0^{\pi/2} \int_0^{+\infty} \left\{ \frac{1}{2} \log(z) - \log[\cos(\tau)] \right\} g^2(z) dz d\tau.$$

We recognize the term $K_\star$ with g instead of f. Based on this remark and Equation (4.4), we get

$$L_\star = \int_0^{+\infty} \omega(z) g^2(z) dz, \tag{4.5}$$

where ω is defined in Equation (4.1).

Combining Equations (4.2), (4.3), (4.4) and (4.5) and uniformizing the notations, we get

$$\int_0^{+\infty}\int_0^{+\infty}\frac{1}{x+y}\left|\log\left(\frac{1}{x+y}\right)\right|f(x)g(y)dxdy$$
$$\leq 4\sqrt{\int_0^{+\infty}\omega(x)f^2(x)dx}\sqrt{\int_0^{+\infty}\omega(x)g^2(x)dx}.$$

The proof of Theorem 4.1 ends. ∎

We complete this theorem with several related results. These are given in the form of propositions.

4.2. Derived propositions

The proposition below shows lower and upper bounds for a specific logarithmic integral inequality.

Proposition 4.2. *Let $f, g : (0, +\infty) \to (0, +\infty)$ be two functions. Assuming that the integrals involved converge, we have*

$$\int_0^{+\infty}\int_0^{+\infty}\frac{1}{x+y}\log\left(\frac{1}{x+y}\right)f(x)g(y)dxdy$$
$$\leq 4\sqrt{\int_0^{+\infty}\omega(x)f^2(x)dx}\sqrt{\int_0^{+\infty}\omega(x)g^2(x)dx}$$

and

$$\int_0^{+\infty}\int_0^{+\infty}\frac{1}{x+y}\log\left(\frac{1}{x+y}\right)f(x)g(y)dxdy$$
$$\geq -4\sqrt{\int_0^{+\infty}\omega(x)f^2(x)dx}\sqrt{\int_0^{+\infty}\omega(x)g^2(x)dx},$$

where ω is given in Equation (4.1).

Proof. The Jesen inequality gives

$$\left|\int_0^{+\infty}\int_0^{+\infty}\frac{1}{x+y}\log\left(\frac{1}{x+y}\right)f(x)g(y)dxdy\right|$$
$$\leq \int_0^{+\infty}\int_0^{+\infty}\frac{1}{x+y}\left|\log\left(\frac{1}{x+y}\right)\right|f(x)g(y)dxdy.$$

We conclude by applying the second inequality in Theorem 4.1. ∎

In the spirit of Propositions 2.6 and 3.3, with some mathematical effort, adjustable parameters can be introduced into the result of Theorem 3.1, as shown in the proposition below.

Proposition 4.3. *Let $\alpha, \beta > 0$ and $f, g : (0, +\infty) \to (0, +\infty)$ be two functions. Assuming that the integrals involved converge, we have*

$$\int_0^{+\infty} \int_0^{+\infty} \frac{1}{x+y} \left| \log\left[\frac{\alpha^\beta}{(x+y)^\beta}\right] \right| f(x)g(y)dxdy$$
$$\leq 4\beta \sqrt{\int_0^{+\infty} \omega\left(\frac{x}{\alpha}\right) f^2(x)dx} \sqrt{\int_0^{+\infty} \omega\left(\frac{x}{\alpha}\right) g^2(x)dx},$$

where ω is given in Equation (4.1).

Proof. We notice that

$$\int_0^{+\infty} \int_0^{+\infty} \frac{1}{x+y} \left| \log\left[\frac{\alpha^\beta}{(x+y)^\beta}\right] \right| f(x)g(y)dxdy$$
$$= \beta \int_0^{+\infty} \int_0^{+\infty} \frac{1}{x+y} \left| \log\left(\frac{\alpha}{x+y}\right) \right| f(x)g(y)dxdy. \tag{4.6}$$

The change of variables $(x, y) = (\alpha u, \alpha v)$ gives

$$\int_0^{+\infty} \int_0^{+\infty} \frac{1}{\alpha(u+v)} \left| \log\left[\frac{\alpha}{\alpha(u+v)}\right] \right| f(\alpha u)g(\alpha v)\alpha^2 dudv$$
$$= \alpha \int_0^{+\infty} \int_0^{+\infty} \frac{1}{u+v} \left| \log\left(\frac{1}{u+v}\right) \right| f_\star(u)g_\star(v)dudv, \tag{4.7}$$

where $f_\star(u) = f(\alpha u)$ and $g_\star(v) = g(\alpha v)$. Applying Theorem 4.1 with $f_\star$ and $g_\star$, and the "backward" change of variables $u = w/\alpha$ and $v = z/\alpha$, we obtain

$$\int_0^{+\infty} \int_0^{+\infty} \frac{1}{u+v} \left| \log\left(\frac{1}{u+v}\right) \right| f_\star(u)g_\star(v)dudv$$
$$\leq 4\sqrt{\int_0^{+\infty} \omega(u) f_\star^2(u)du} \sqrt{\int_0^{+\infty} \omega(v) g_\star^2(v)dv}$$
$$= 4\sqrt{\int_0^{+\infty} \omega\left(\frac{w}{\alpha}\right) f_\star^2\left(\frac{w}{\alpha}\right) \frac{1}{\alpha} dw} \sqrt{\int_0^{+\infty} \omega\left(\frac{z}{\alpha}\right) g_\star^2\left(\frac{z}{\alpha}\right) \frac{1}{\alpha} dz}$$
$$= \frac{4}{\alpha} \sqrt{\int_0^{+\infty} \omega\left(\frac{x}{\alpha}\right) f^2(x)dx} \sqrt{\int_0^{+\infty} \omega\left(\frac{x}{\alpha}\right) g^2(x)dx}. \tag{4.8}$$

Combining Equations (4.6), (4.7) and (4.8) together, we conclude the proof of Proposition 4.3. ■

Similar to Propositions 2.7 and 3.4, the result below offers another way of looking at Theorem 4.1.

Proposition 4.4. *Let $f, g : (0, +\infty) \to (0, +\infty)$ be two functions. Assuming that the integrals involved converge, we have*

$$\int_0^{+\infty} \int_0^{+\infty} \frac{1}{xy(x+y)} \left| \log\left(\frac{xy}{x+y}\right) \right| f(x)g(y)dxdy$$
$$\leq 4\sqrt{\int_0^{+\infty} \frac{1}{x^2}\omega\left(\frac{1}{x}\right) f^2(x)dx} \sqrt{\int_0^{+\infty} \frac{1}{x^2}\omega\left(\frac{1}{x}\right) g^2(x)dx},$$

where ω is given in Equation (4.1).

Proof. Using the change of variables $(x, y) = (1/u, 1/v)$, we get

$$\begin{aligned}
&\int_0^{+\infty}\int_0^{+\infty} \frac{1}{xy(x+y)}\left|\log\left(\frac{xy}{x+y}\right)\right| f(x)g(y)dxdy \\
&= \int_0^{+\infty}\int_0^{+\infty} \frac{1}{1/x+1/y}\left|\log\left(\frac{1}{1/x+1/y}\right)\right| f(x)g(y)\left(-\frac{1}{x^2}\right)\left(-\frac{1}{y^2}\right)dxdy \\
&= \int_0^{+\infty}\int_0^{+\infty} \frac{1}{u+v}\left|\log\left(\frac{1}{u+v}\right)\right| f_\dagger(u)g_\dagger(v)dudv, \qquad (4.9)
\end{aligned}$$

where $f_\dagger(u) = f(1/u)$ and $g_\dagger(v) = g(1/v)$. Based on Theorem 3.1 with $f_\dagger$ and $g_\dagger$, and the “backward” change of variables $u = 1/w$ and $v = 1/z$, we get

$$\begin{aligned}
&\int_0^{+\infty}\int_0^{+\infty} \frac{1}{u+v}\left|\log\left(\frac{1}{u+v}\right)\right| f_\dagger(u)g_\dagger(v)dudv \\
&\le 4\sqrt{\int_0^{+\infty}\omega(u)f_\dagger^2(u)du}\sqrt{\int_0^{+\infty}\omega(v)g_\dagger^2(v)dv} \\
&= 4\sqrt{\int_0^{+\infty}\omega\left(\frac{1}{w}\right)f_\dagger^2\left(\frac{1}{w}\right)\frac{1}{w^2}dw}\sqrt{\int_0^{+\infty}\omega\left(\frac{1}{z}\right)g_\dagger^2\left(\frac{1}{z}\right)\frac{1}{z^2}dz} \\
&= 4\sqrt{\int_0^{+\infty}\frac{1}{x^2}\omega\left(\frac{1}{x}\right)f^2(x)dx}\sqrt{\int_0^{+\infty}\frac{1}{x^2}\omega\left(\frac{1}{x}\right)g^2(x)dx}. \qquad (4.10)
\end{aligned}$$

The proof ends by combining Equations (4.9) and (4.10). ■

5. Conclusion and perspectives

In this paper, we have studied a special logarithmic integral inequality, which can be seen as a variant of the Hilbert integral inequality. It is special in the sense that it depends on a particular logarithmic non-homogeneous kernel. Despite its obvious mathematical appeal, it has not received much attention in the literature. The proof was carried out using several intermediate results, which may be of independent interest. Among these results, the Lobachevskii function appears, showing a new application of this function in analysis.

One possible approach to this work is to provide a rigorous proof of the sharpness of the demonstrated inequalities. Another possibility is the study of parametric versions of the main double integral. For instance, for any $a, b \in (0, 1)$, we can consider

$$\int\int_{\{(x,y)\in(0,+\infty)^2;\ x+y\le 1\}} \frac{1}{x+y}\log\left[\frac{\max(a,b)}{ax+by}\right] f(x)g(y)dxdy,$$

or

$$\int_0^{+\infty}\int_0^{+\infty} \frac{1}{x+y}\left|\log\left[\frac{\max(a,b)}{ax+by}\right]\right| f(x)g(y)dxdy.$$

In this context, further work is needed to manage the parameters a and b for sharp upper bounds. The techniques developed in our proofs may also be inspiring in other contexts of integral inequalities. We leave this work for the future.

Acknowledgements

The author would like to thank the associate editor and the reviewer for their constructive comments on the paper.

References

[1] V. Adiyasuren, T. Batbold, M. Krnić, Hilbert-type inequalities involving differential operators, the best constants and applications, *Math. Inequal. Appl.*, **18** (2015) 111–124.

[2] Q. Chen, B. C. Yang, A survey on the study of Hilbert-type inequalities, *J. Inequal. Appl.*, **2015** (2015) 1–30.

[3] C. Chesneau, A three-parameter logarithmic generalization of the Hilbert integral inequality, *J. Math. Anal. Model.*, **6(1)** (2025) 68–81.

[4] C. Chesneau, General inequalities of the Hilbert integral type using the method of switching to polar coordinates, *Hilbert J. Math. Anal.*, **3** (2024) 7–26.

[5] I. S. Gradshteyn, I. M. Ryzhik, *Table of Integrals, Series, and Products*, 7th Edition, Academic Press, Burlington, 2007.

[6] G. H. Hardy, J. E. Littlewood, G. Pólya, *Inequalities*, Cambridge University Press, Cambridge, 1934.

[7] Y. Li, Y. Qian, B. He, On further analogs of Hilbert's inequality, *Int. J. Math. Math. Sci.*, **2007** (2007) 1–6.

[8] N. I. Lobachevskii, *Polnoye sobraniye sochineniy (Complete Works)*, Vol. III, Gostekhizdat, Moscow and Leningrad, 1946–1951.

[9] D. C. Ullrich, A simple elementary proof of Hilbert's inequality, *Amer. Math. Monthly*, **120(2)** (2013) 161–164.

[10] D. M. Xin, A Hilbert-type integral inequality with the homogeneous kernel of zero degree, *Math. Theory Appl.*, **30(2)** (2010) 70–74.

[11] J. S. Xu, Hardy-Hilbert's inequalities with two parameters, *Adv. Math.*, **36(2)** (2007) 63–76.

[12] B. C. Yang, On Hilbert's integral inequality, *J. Math. Anal. Appl.*, **220(2)** (1998) 778–785.

[13] B. C. Yang, On a basic Hilbert-type inequality, *J. Guangdong Educ. Inst.*, **26(3)** (2006) 1–5.

[14] B. C. Yang, *Hilbert-Type Integral Inequalities*, Bentham Science Publishers, The United Arab Emirates, 2009.

[15] B. C. Yang, Hilbert-type integral inequality with non-homogeneous kernel, *J. Shanghai Univ.*, **17** (2011) 603–605.

Appendix: On the Lobachevskii function

In this part, which can be considered independent of the main study, some facts about the Lobachevskii function are recalled. First of all, it was introduced by N. I. Lobachevskii in 1829 with reference to [8]. It is mathematically defined by

$$\mathcal{L}(x) := -\int_0^x \log[\cos(t)]dt,$$

with $x \in [-\pi/2, \pi/2]$. A list of the notable features of this function is given below.

- Some remarkable expressions of the Lobachevskii function at certain values are as follows:

$$\mathcal{L}(0) = 0, \quad \mathcal{L}\left(\frac{\pi}{2}\right) = \frac{\pi}{2}\log(2), \quad \mathcal{L}\left(\frac{\pi}{4}\right) = \frac{1}{4}\left[\pi\log(2) - 2G\right],$$

where G denotes the Catalan constant, i.e., $G = \sum_{k=0}^{+\infty}(-1)^k/(2k+1)^2 \approx 0.915966$.
- Some approximated values of the Lobachevskii function at certain values of the form π/m, where m denotes a positive integer, are as follows:

$$\mathcal{L}\left(\frac{\pi}{2}\right) \approx 1.08879, \quad \mathcal{L}\left(\frac{\pi}{3}\right) \approx 0.218391, \quad \mathcal{L}\left(\frac{\pi}{4}\right) \approx 0.0864137,$$

$$\mathcal{L}\left(\frac{\pi}{5}\right) \approx 0.0431093, \quad \mathcal{L}\left(\frac{\pi}{6}\right) \approx 0.0246171, \quad \mathcal{L}\left(\frac{\pi}{7}\right) \approx 0.0153819,$$

$$\mathcal{L}\left(\frac{\pi}{8}\right) \approx 0.0102536, \quad \mathcal{L}\left(\frac{\pi}{9}\right) \approx 0.00717721, \quad \mathcal{L}\left(\frac{\pi}{10}\right) \approx 0.0052197,$$

$$\mathcal{L}\left(\frac{\pi}{11}\right) \approx 0.00391475, \quad \mathcal{L}\left(\frac{\pi}{12}\right) \approx 0.00301134, \quad \mathcal{L}\left(\frac{\pi}{13}\right) \approx 0.00236606,$$

$$\mathcal{L}\left(\frac{\pi}{14}\right) \approx 0.00189285, \quad \mathcal{L}\left(\frac{\pi}{15}\right) \approx 0.00153795, \quad \mathcal{L}\left(\frac{\pi}{16}\right) \approx 0.00126655.$$

- The Lobachevskii function is an odd function, i.e., for any $x \in [-\pi/2, \pi/2]$, we have

$$\mathcal{L}(-x) = -\mathcal{L}(x).$$

- Its derivative is given as

$$\mathcal{L}'(x) = -\log[\cos(x)].$$

- For any $x \in [-\pi/2, \pi/2]$, we also have

$$\mathcal{L}(\pi \pm x) = \pi\log(2) \pm \mathcal{L}(x).$$

- For any $x \in [0, \pi/4)$, the following functional equation holds:

$$\mathcal{L}(x) - \mathcal{L}\left(\frac{\pi}{2} - x\right) = \left(x - \frac{\pi}{4}\right)\log(2) - \frac{1}{2}\mathcal{L}\left(\frac{\pi}{2} - 2x\right).$$

- The Lobachevskii function has the following series representation:

$$\mathcal{L}(x) = x\log(2) + \frac{1}{2}\sum_{k=1}^{+\infty}(-1)^k\frac{\sin(2kx)}{k^2}.$$

The Lobachevskii function naturally appears in some integral formulas. The most concise of them are presented below.

- We have

$$\int_0^{+\infty} \frac{x}{2\cosh(x)-1}dx = \frac{4}{\sqrt{3}}\left[\frac{\pi}{3}\log(2) - \mathcal{L}\left(\frac{\pi}{3}\right)\right],$$

where $\cosh(x) = (e^x + e^{-x})/2$.

- We have

$$\int_0^{+\infty} \frac{x}{\cosh(2x)+\cos(2t)}dx = \frac{1}{\sin(2t)}\left[t\log(2) - \mathcal{L}(t)\right],$$

for t not equal to an integer multiplied by $\pi/2$.

- We have

$$\int_0^{+\infty} \frac{x\cosh(x)}{\cosh(2x)-\cos(2t)}dx = \frac{1}{\sin(t)}\left[\frac{\pi}{2}\log(2) - \mathcal{L}\left(\frac{t}{2}\right) - \mathcal{L}\left(\frac{\pi-t}{2}\right)\right],$$

for t not equal to an integer multiplied by π.

- For any $u \in [0, \pi]$, we have

$$\int_0^u \log[\sin(x)]dx = \mathcal{L}\left(\frac{\pi}{2} - u\right) - \mathcal{L}\left(\frac{\pi}{2}\right).$$

For any $u \in [0, \pi/2]$, we have

$$\int_0^u \log[\tan(x)]dx = \mathcal{L}(u) + \mathcal{L}\left(\frac{\pi}{2} - u\right) - \mathcal{L}\left(\frac{\pi}{2}\right).$$

- We have

$$\int_0^{\pi/2} \log\left[\sin(t)\sin(x) + \sqrt{1-\cos^2(t)\sin^2(x)}\right]dx$$
$$= \frac{\pi}{2}\log(2) - 2\mathcal{L}\left(\frac{t}{2}\right) - 2\mathcal{L}\left(\frac{\pi-t}{2}\right).$$

- We have

$$\int_0^{\pi/4} \frac{\log\left[\tan(x)\right]\sin(2x)}{1-\cos^2(t)\sin^2(2x)}dx$$
$$= \frac{1}{\sin(2t)}\left[\mathcal{L}\left(\frac{\pi}{2} - t\right) - \left(\frac{\pi}{2} - t\right)\log(2)\right],$$

for t not equal to an integer multiplied by $\pi/2$.

- For any $\alpha \in (0, \pi/2)$, we have

$$\int_0^{\pi/2} \frac{\log\left[1+\cos(\alpha)\cos(x)\right]\cos(x)}{1-\cos^2(\alpha)\cos^2(x)}dx$$
$$= \frac{1}{\sin(\alpha)\cos(\alpha)}\left[\mathcal{L}\left(\frac{\pi}{2} - \alpha\right) - \alpha\log[\sin(\alpha)]\right]$$

and, the "sister" formula,

$$\int_0^{\pi/2} \frac{\log[1-\cos(\alpha)\cos(x)]\cos(x)}{1-\cos^2(\alpha)\cos^2(x)}dx$$
$$= \frac{1}{\sin(\alpha)\cos(\alpha)}\left[\mathcal{L}\left(\frac{\pi}{2}-\alpha\right)+(\pi-\alpha)\log[\sin(\alpha)]\right].$$

Further integral formulas can be found in [5, 8]. We end this brief overview by recalling how the Lobachevskii function is used as a powerful tool in various mathematical analyses.