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An F-norm on sequences spaces

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Abstract On sequences spaces, we can choose a Young function that plays a role in determining their norm structure. In this article, we modify the Young function by replacing its convexity property with concavity to define the F-norm in these spaces. Furthermore, we explore the properties of the modified Young function. Additionally, we investigate the completeness of the space, allowing it to be classified as an F-space (Fréchet space).

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1. Introduction

The concept of a norm serves as a crucial tool for measuring the "size" or "magnitude" of vectors within a given vector space. Just as we intuitively understand the length of a line segment in classical Euclidean geometry, a norm extends this idea to more abstract spaces, providing a consistent way to quantify how large a vector is, regardless of its direction. Norms play a crucial role in linear algebra and functional analysis, as they provide a way to measure distances. By providing a way to measure distances, norms enable the study of convergence, continuity, and functional behavior in both finite and infinite-dimensional spaces. Their diverse applications across mathematics, physics, computer science, and engineering make them one of the most important concepts in modern analysis.

Let V be a real vector space. See in [15, 25] that formally, a norm is a mapping $\|\cdot\|_V : V \to \mathbb{R}$ with the following properties

(1) $\|x\|_V \geq 0$, for every $x \in V$; $\|x\|_V = 0$ if and only if $x = 0 \in V$;
(2) $\|\alpha x\|_V = |\alpha| \, \|x\|_V$, for every $x \in V$ and for every scalar $\alpha \in \mathbb{R}$;
(3) $\|x + y\|_V \leq \|x\|_V + \|y\|_V$ for every $x, y \in V$.

We have a pair of $(V, \|\cdot\|_V)$, which is called a normed space.

In the above, a norm must satisfy several conditions, one of which is absolute homogeneity $\|\alpha x\|_V = |\alpha| \cdot \|x\|_V$ for every $x \in V$ and scalars α. A modification of this

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property, considered "weaker", was put forward by the French mathematician René Maurice Fréchet, resulting in what is now called the "F-norm" (Fréchet norm). Let V be a real vector space. A function $\|\cdot\|_{FV} : V \to \mathbb{R}$ is called an F-norm if it satisfies the following four properties

(1) $\|x\|_{FV} \geq 0$, for every $x \in V$, $\|x\|_{FV} = 0$ if and only if $x = 0 \in V$,
(2) $\|-x\|_{FV} = \|x\|_{FV}$, for every $x \in V$. This means that the norm does not depend on the direction of the vector (flipping the sign does not change the magnitude),
(3) $\|x + y\|_{FV} \leq \|x\|_{FV} + \|y\|_{FV}$ for every $x, y \in V$,
(4) If (α_n) is a sequence of scalars converging to α, and (x_n) is a sequence in V such that $\|x_n - x\|_{FV} \to 0$ as $n \to \infty$, then we must have $\|\alpha_n x_n - \alpha x\|_{FV} \to 0$ as $n \to \infty$.

A pair of $(V, \|\cdot\|_{FV})$ is called an F-normed space. The properties (2) and (4) are conditions serve as a key distinction between an F-norm and a norm. In essence, while normed spaces require strict proportionality under scalar multiplication, F-normed spaces only require that this property "holds in the limit for sequences". Thus, since every norm satisfies absolute homogeneity exactly, it automatically satisfies this weaker sequential condition as well. This means that every norm automatically qualifies as an F-norm.

One important property of normed spaces is completeness. It means that if we have a sequence of vectors that keep getting closer and closer together (we call it a Cauchy sequence), then there must be a final vector in the space that they all approach. A normed space that is complete is called a Banach space, named after the Polish mathematician Stefan Banach. There is also a more general type of space called an F-space (Fréchet space). This is a complete space like a Banach space, but it doesn't always use a norm to measure size, maybe it uses a different kind of distance function (called a metric). So, an F-normed space is called an F-space if it is complete, that is, if every Cauchy sequence converges with respect to the F-norm to a point within the space. Several studies on F-norms can be found in [3, 27, 28].

This paper examines sequence spaces defined using a criterion based on the Young function, referred to as discrete Orlicz spaces (see [1, 5, 10, 18, 26]). It is well established that these spaces are normed spaces endowed with the Luxemburg norm, which also makes them F-normed spaces. In this study, we replace the convexity property of the Young function with a concave property, introducing what we call the M-Young function (Young modification). Sequences spaces satisfying the M-Young function criterion are then equipped with an F-norm, classifying them as F-normed spaces. Our analysis focuses on the behavior of sequences within these spaces, particularly Cauchy sequences with respect to the F-norm. We demonstrate that every Cauchy sequence respect to converges to a limit that remains within the space, thereby confirming the completeness of the sequences space with the F-norm.

2. Young Functions and Sequences Spaces

Young functions are central to the theory of Orlicz spaces, providing the mathematical foundation for their structure and fundamental properties. Orlicz spaces were formally introduced in 1931 [2, 19] by Polish mathematicians Zygmunt Wilhelm Birnbaum and Władysław Orlicz as a broad generalization of classical Lebesgue spaces. The theory was further advanced through significant contributions by Malempati Madhusudana Rao (see [22, 23]). Additionally, in 1961, Robert Welland investigated the inclusion relations among Orlicz spaces [30].

In L^p spaces, it is known that the function x^p can be substituted by a more general convex function Φ, known as an N-function, which leads to the study of the corresponding Orlicz space. The first thorough analysis of Orlicz spaces, treating Φ as an N-function (Nice Young function), was provided by Krasnoseĺskii and Rutickii (1961), as referenced in [14, 16, 20]. Many researchers refer to $\Phi(t)$ as an N-Young function due to its excellent properties that simplify mathematical analysis. However, in this context, we will simply call it a Young function. Let $\Phi(t)$ be a function defined for $0 \leq t < \infty$. The function $\Phi(t)$ is called a Young function if it meets the following criteria

(1) (Convexity) The function $\Phi(t)$ is convex, meaning that for any $0 \leq \lambda \leq 1$, the inequality $\Phi(\lambda x + (1-\lambda)y) \leq \lambda\Phi(x) + (1-\lambda)\Phi(y)$ holds for every $0 \leq x, y < \infty$,
(2) (Monotonicity) The function is strictly increasing, $\Phi(x) < \Phi(y)$ for every $x < y$,
(3) (Zero at the origin) The function starts at zero, meaning $\Phi(0) = 0$,
(4) (Unbounded Growth) The function satisfies $\Phi(t) \to \infty \quad \text{as } t \to \infty$.

Now, since we are dealing with sequence spaces over the real numbers, we will sometimes use the summation notation $\sum\limits_k$ instead of $\sum\limits_{k=1}^{\infty}$ for simplicity. In the study of sequence spaces ℓ^p for $p \geq 1$, where sequences satisfy $\sum\limits_k |x_k|^p < \infty$, a fundamental function to consider is $f(t) = t^p$, for $0 \leq t < \infty$. This function is essential in characterizing the properties of sequences in ℓ^p. Specifically, the norm of a sequence $x = (x_k)$ is given by

$$\|x\|_p := \left(\sum_k |x_k|^p \right)^{\frac{1}{p}}.$$

Some researches on p-summable sequence spaces is available in [4, 7, 9, 11–13]. Here, f ensures that the summation inside the norm is well-defined and appropriately scales each term according to p. Readers can verify on their own as an exercise that f is a convex function, strictly increasing, zero at the origin, and exhibits unbounded growth as $t \to \infty$. Since f satisfies all these conditions, it qualifies as a Young function.

Considering Φ as a general Young function, we define the set ℓ^Φ as the collection of sequences (x_k) satisfying

$$\ell^\Phi := \left\{ (x_k) \middle| \sum_k \Phi\left(\frac{|x_k|}{A} \right) < \infty \right\},$$

for some $A > 0$. This set, known as a discrete Orlicz space, forms a vector space. However, defining a norm on ℓ^Φ presents a challenge because Φ is generally not homogeneous, making conventional norm definitions inapplicable. To address this, we employ an alternative approach called the Luxemburg norm, given by

$$\|x\|_\Phi := \inf \left\{ A > 0 \;\middle|\; \sum_k \Phi\left(\frac{|x_k|}{A} \right) \leq 1 \right\},$$

for every $x \in \ell^\Phi$. Refer to [1, 5, 18, 21, 26] for the verification that $\|\cdot\|_\Phi$ qualifies as a norm. Consequently, $\left(\ell^\Phi, \|\cdot\|_\Phi\right)$ is recognized as a normed space. Further developments and generalizations of the discrete Orlicz space are available in [6, 17]. The convexity property of Φ plays a crucial role in proving that $\|\cdot\|_\Phi$ satisfies the triangle inequality.

The convexity of Φ has been discussed in more detail in [8]. Moreover, $(\ell^{\Phi}, \|\cdot\|_{\Phi})$ as a Banach space is presented in [10]. As a result, $\big(\ell^{\Phi}, \|\cdot\|_{\Phi}\big)$ also qualifies as an F-space, since $\|\cdot\|_{\Phi}$ can be interpreted as an F-norm.

3. M-Young Functions and Sequences Spaces

Now, we modify the Young function by replacing the convexity property with concavity. This modified function is called the M-Young function, denoted by $\widehat{\Phi}$. A similar approach can also be found in [29, 31]. We will investigate the concave properties of the M-Young function.

Lemma 3.1. *Let $\widehat{\Phi}$ be an M-Young function. We have*

$$\frac{\widehat{\Phi}(w)-\widehat{\Phi}(v)}{w-v} \leq \frac{\widehat{\Phi}(w)-\widehat{\Phi}(u)}{w-u} \leq \frac{\widehat{\Phi}(v)-\widehat{\Phi}(u)}{v-u}$$

for every $0 \leq u < v < w < \infty$.

Proof. Since $\widehat{\Phi}$ is a concave function, it satisfies the inequality

$$\widehat{\Phi}(\lambda u + (1-\lambda)w) \geq \lambda\widehat{\Phi}(u) + (1-\lambda)\widehat{\Phi}(w)$$

for every $u, w \in [0, \infty)$ and $0 \leq \lambda \leq 1$. Setting $\lambda = \frac{w-v}{w-u}$ for $u < v < w$, we obtain

$$\widehat{\Phi}(v) \geq \frac{w-v}{w-u}\widehat{\Phi}(u) + \frac{v-u}{w-u}\widehat{\Phi}(w). \tag{3.1}$$

Subtracting $\widehat{\Phi}(u)$ from both sides of (3.1), we derive

$$\widehat{\Phi}(v) - \widehat{\Phi}(u) \geq \frac{v-u}{w-u}\big(\widehat{\Phi}(w) - \widehat{\Phi}(u)\big),$$

$$\frac{\widehat{\Phi}(v)-\widehat{\Phi}(u)}{v-u} \geq \frac{\widehat{\Phi}(w)-\widehat{\Phi}(u)}{w-u}.$$

Similarly, subtracting $\widehat{\Phi}(w)$ from both sides of (3.1) gives

$$\widehat{\Phi}(v) - \widehat{\Phi}(w) \geq \frac{w-v}{w-u}\big(\widehat{\Phi}(u) - \widehat{\Phi}(w)\big),$$

$$\frac{\widehat{\Phi}(w)-\widehat{\Phi}(u)}{w-u} \geq \frac{\widehat{\Phi}(w)-\widehat{\Phi}(v)}{w-v}.$$

As a result, we establish the inequality

$$\frac{\widehat{\Phi}(w)-\widehat{\Phi}(v)}{w-v} \leq \frac{\widehat{\Phi}(w)-\widehat{\Phi}(u)}{w-u} \leq \frac{\widehat{\Phi}(v)-\widehat{\Phi}(u)}{v-u},$$

for every $u < v < w$. ∎

By Lemma 3.1, take $u = 0$, we have a special case

$$\frac{\widehat{\Phi}(w)-\widehat{\Phi}(v)}{w-v} \leq \frac{\widehat{\Phi}(w)}{w} \leq \frac{\widehat{\Phi}(v)}{v}, \tag{3.2}$$

for every $0 < v < w < \infty$.

The concavity of the M-Young function ensures that as the input value increases, the function's growth slows down in a controlled manner. This is reflected in the given inequalities, which show that the difference quotients, representing the function's average rate of change, decrease as we move to larger values.

The function $\widehat{\Phi}$ may not have a simple form, which makes it important to analyze and compare the growth rates of complexity functions. To facilitate these comparisons, we use Big-O notation, introduced by the German mathematician Paul Bachmann in 1894. This notation was later popularized by Edmund Landau and is now widely known as Bachmann-Landau notation or asymptotic notation. It describes how functions behave as their inputs approach very small or very large values. Since its introduction, Big-O notation has become a widely adopted tool in algorithm analysis, helping researchers to precisely characterize and analyze the growth rates of complexity functions [24]. Without loss of generality, consider two functions $f, g : [0, \infty) \to [0, \infty)$ such that

$$\lim_{t\to 0} f(t) = \lim_{t\to 0} g(t) = 0 \qquad \text{and} \qquad \lim_{t\to\infty} f(t) = \lim_{t\to\infty} g(t) = \infty.$$

We say $f(t) = O(g(t))$ if there exists a constant $C_1 > 0$ such that

$$\lim_{t\to 0} \frac{f(t)}{g(t)} = C_1.$$

Alternatively, using a different condition, we say $f(t) = O(g(t))$ if there exists a constant $C_2 > 0$ such that

$$\lim_{t\to\infty} \frac{f(t)}{g(t)} = C_2.$$

Lemma 3.2. *Let $\widehat{\Phi}$ be an M-Young function. There exists $0 < \gamma \leq 1$ such that*

$$\widehat{\Phi}(t) \approx O(t^\gamma)$$

for every $t \in (0, 1)$.

Proof. Since $\widehat{\Phi}$ is an M-Young function, then we have (3.2). So the condition $\frac{\widehat{\Phi}(w)}{w} \leq \frac{\widehat{\Phi}(v)}{v}$, for every $v < w$, implies that the function $h(v) := \frac{\widehat{\Phi}(v)}{v}$ is non-increasing. Now setting $w = 1$, we obtain $\widehat{\Phi}(1)v \leq \widehat{\Phi}(v)$, for every $v \in (0, 1]$. Since $\widehat{\Phi}$ is an increasing and not a constant function, then see the following that $\widehat{\Phi}(t)$ decays more slowly than t.

$$\widehat{\Phi}(1)t \leq \widehat{\Phi}(t) < \widehat{\Phi}(1) \qquad \text{or} \qquad t \leq P(t) < 1 \quad \text{with} \quad P(t) := \frac{\widehat{\Phi}(t)}{\widehat{\Phi}(1)}$$

for every $t \in (0, 1)$. Apply a logarithm to convert the above into

$$-\ln(t) \geq -\ln(P(t)) > 0 \qquad \text{or} \qquad 0 < \frac{\ln(P(t))}{\ln(t)} \leq 1.$$

For $\lim\limits_{t\to 0} \frac{\ln(P(t))}{\ln(t)} > 0$, there exists $\gamma \in (0, 1]$ such that

$$\lim_{t\to 0} \frac{\ln(P(t))}{\ln(t)} = \gamma.$$

It means that $\left|\frac{\ln(P(t))}{\ln(t)} - \gamma\right| \leq \beta$, for every $\beta \in (0, \gamma)$ and $t_\beta \geq t > 0$ (t depends on β). This implies that

$$t^{\gamma+\beta} \leq P(t) = \frac{\widehat{\Phi}(t)}{\widehat{\Phi}(1)} \leq t^{\gamma-\beta}.$$

Therefore, we can conclude that $\widehat{\Phi}(t) \approx O(t^\gamma)$ for every $t \in (0, 1)$. ∎

Lemma 3.2 implies that M-Young functions decay at a controlled rate near zero, ensuring they do not decrease too rapidly. The condition helps maintain useful mathematical properties, particularly in estimating small values of the function. We provide examples of M-Young functions that meet this condition, such as (1) $\widehat{\Phi}(t) = t^\alpha$ with $0 < \alpha < 1$ for every $t \geq 0$, and (2) $\widehat{\Phi}(t) = \log(1+t)$ for every $t \geq 0$. Moreover, it can be verified that $\log(1+t)$ behaves like $O(t)$ for every $t \in (0,1)$. These functions illustrate how the given bound holds, confirming that their rate of decay is controlled within the specified range.

Lemma 3.3. *Let $\widehat{\Phi}$ be an M-Young function. For $0 \leq v, w < \infty$, we have*

$$\widehat{\Phi}(v+w) \leq \widehat{\Phi}(v) + \widehat{\Phi}(w).$$

Proof. Suppose $\widehat{\Phi}$ is an M-Young function. Consider the following cases

(1) For $v = 0$ or $w = 0$. It is clear that the inequality

$$\widehat{\Phi}(v+w) \leq \widehat{\Phi}(v) + \widehat{\Phi}(w)$$

holds trivially.

(2) For $v \neq 0$ and $w \neq 0$. We can state $v < v + w$ and $w < v + w$, so we utilize inequality (3.2) to obtain

$$\frac{\widehat{\Phi}(v)}{v} \geq \frac{\widehat{\Phi}(v+w)}{v+w}, \quad \frac{\widehat{\Phi}(w)}{w} \geq \frac{\widehat{\Phi}(v+w)}{v+w}.$$

Consequently, we also have

$$(v+w)\widehat{\Phi}(v) \geq v\widehat{\Phi}(v+w), \quad (v+w)\widehat{\Phi}(w) \geq w\widehat{\Phi}(v+w).$$

Adding both sides of these inequalities results in

$$(v+w)\big(\widehat{\Phi}(v) + \widehat{\Phi}(w)\big) \geq (v+w)\widehat{\Phi}(v+w).$$

Thus, we conclude that $\widehat{\Phi}(v+w) \leq \widehat{\Phi}(v) + \widehat{\Phi}(w)$. ■

Notice that Lemma 3.3 is essential as it ensures that the M-Young function $\widehat{\Phi}$ does not grow too rapidly when adding two values. This property helps regulate the function's behavior when summing over a sequence, preventing uncontrolled growth.

Using $\widehat{\Phi}$ as an M-Young function, we now define the set $\ell^{\widehat{\Phi}}$ as the collection of sequences (x_k) satisfying

$$\ell^{\widehat{\Phi}} := \left\{ (x_k) \;\middle|\; \sum_k \widehat{\Phi}\left(|x_k|\right) < \infty \right\}.$$

We also define a mapping $\|\cdot\|_{F\widehat{\Phi}} : \ell^{\widehat{\Phi}} \to \mathbb{R}$ by

$$\|x\|_{F\widehat{\Phi}} := \sum_k \widehat{\Phi}\left(|x_k|\right),$$

for every $x \in \ell^{\widehat{\Phi}}$. The next theorem demonstrates that $\|\cdot\|_{F\widehat{\Phi}}$ defines an F-norm on $\ell^{\widehat{\Phi}}$. The proof relies significantly on Lemmas 3.2 and 3.3. In particular, Lemma 3.3 clearly ensures that $\|\cdot\|_{F\widehat{\Phi}}$, which measures the size of a sequence, satisfies the triangle inequality.

Theorem 3.4. *The mapping $\|\cdot\|_{F\widehat{\Phi}}$ on $\ell^{\widehat{\Phi}}$ is an F-norm.*

Proof. We will show that $\|\cdot\|_{F\widehat{\Phi}}$ satisfies four properties of F-norm.

(1) (**non-negativity**) Since $\widehat{\Phi}$ is an M-Young function, then it is assumed to be non-negative and satisfies $\widehat{\Phi}(t) \geq 0$ for all $t \geq 0$. Since $|x_k|$ is also non-negative, then each term in the summation is non-negative $\widehat{\Phi}(|x_k|) \geq 0$ for every $k \in \mathbb{N}$. Consequently,

$$\|x\|_{F\widehat{\Phi}} = \sum_k \widehat{\Phi}(|x_k|) \geq 0.$$

Now we will show that $\|x\|_{F\widehat{\Phi}} = 0$ if and only if $x = 0 \in \ell^{\widehat{\Phi}}$.

- Suppose that $\sum_k \widehat{\Phi}(|x_k|) = 0$. So, each term $\widehat{\Phi}(|x_k|)$ in the summation must be zero. Since $\widehat{\Phi}$ is an M-Young function, then $\widehat{\Phi}$ is strictly positive for every $t > 0$, meaning that

$$\widehat{\Phi}(|x_k|) = 0 \quad \Rightarrow \quad |x_k| = 0.$$

Thus, $x_k = 0$ for for every $k \in \mathbb{N}$, implying that x is the zero sequence.

- Conversely, if $x = 0$, then $|x_k| = 0$ for every $k \in \mathbb{N}$, and thus

$$\|x\|_{F\widehat{\Phi}} = \sum_k \widehat{\Phi}(0) = 0.$$

(2) (**symmetry property**) By the definition $\|x\|_{F\widehat{\Phi}} := \sum_k \widehat{\Phi}(|x_k|)$, now, consider $-x$, which means each component of x is replaced with $(-x)_k = -x_k$ for every $k \in \mathbb{N}$. We get $|-x_k| = |x_k|$. Since $\widehat{\Phi}$ depends only on the magnitude $|x_k|$, then $\widehat{\Phi}(|-x_k|) = \widehat{\Phi}(|x_k|)$. Summing over all indices k, we obtain

$$\|-x\|_{F\widehat{\Phi}} = \sum_k \widehat{\Phi}(|-x_k|) = \sum_k \widehat{\Phi}(|x_k|) = \|x\|_{F\widehat{\Phi}}.$$

(3) (**triangle inequality**) Let $x, y \in \ell^{\widehat{\Phi}}$. We have $\|x+y\|_{F\widehat{\Phi}} = \sum_k \widehat{\Phi}(|x_k + y_k|)$. Using the subadditivity property of $\widehat{\Phi}$, we get

$$\sum_k \widehat{\Phi}(|x_k + y_k|) \leq \sum_k \widehat{\Phi}(|x_k| + |y_k|).$$

Applying Lemma 3.3 for $\widehat{\Phi}$, we further obtain

$$\sum_k \widehat{\Phi}(|x_k| + |y_k|) \leq \sum_k \widehat{\Phi}(|x_k|) + \sum_k \widehat{\Phi}(|y_k|).$$

Thus, we conclude $\|x+y\|_{F\widehat{\Phi}} \leq \|x\|_{F\widehat{\Phi}} + \|y\|_{F\widehat{\Phi}}$.

(4) (**convergence of scalar multiplication**) Suppose that $|\alpha(n) - \alpha| \to 0$ and $\|x(n) - x\|_{F\widehat{\Phi}} \to 0$ as $n \to \infty$. Using triangle inequality property, we get

$$\begin{aligned}
\|\alpha(n)x(n) - \alpha x\|_{F\widehat{\Phi}} &= \|(\alpha(n) - \alpha)x(n) + \alpha(x(n) - x)\|_{F\widehat{\Phi}} \\
&\leq \|(\alpha(n) - \alpha)x(n)\|_{F\widehat{\Phi}} + \|\alpha(x(n) - x)\|_{F\widehat{\Phi}} \\
&= \|(\alpha(n) - \alpha)(x(n) - x) + (\alpha(n) - \alpha)x\|_{F\widehat{\Phi}} \\
&\quad + \|\alpha(x(n) - x)\|_{F\widehat{\Phi}} \\
&\leq \|(\alpha(n) - \alpha)(x(n) - x)\|_{F\widehat{\Phi}} + \|(\alpha(n) - \alpha)x\|_{F\widehat{\Phi}} \\
&\quad + \|\alpha(x(n) - x)\|_{F\widehat{\Phi}}.
\end{aligned}$$

Now, choose a sufficiently large $n_0 \in \mathbb{N}$ such that for every $n > n_0$, the following conditions hold for all $k \in \mathbb{N}$

- $|\alpha(n) - \alpha| < 1$,
- $|x_k(n) - x_k| < 1$,
- $|\alpha(n) - \alpha||x_k| < 1$,
- $|\alpha||x_k(n) - x_k| < 1$.

Using Lemma 3.2, we get

$$\begin{aligned}\widehat{\Phi}(|\alpha(n) - \alpha||x_k(n) - x_k|) &\approx |\alpha(n) - \alpha|^\gamma |x_k(n) - x_k|^\gamma \\ &\approx |\alpha(n) - \alpha|^\gamma \widehat{\Phi}\big(|x_k(n) - x_k|\big),\end{aligned}$$

$$\begin{aligned}\widehat{\Phi}(|\alpha(n) - \alpha||x_k|) &\approx |\alpha(n) - \alpha|^\gamma |x_k|^\gamma \\ &\approx |\alpha(n) - \alpha|^\gamma |1 + \|x\|_{F\widehat{\Phi}}|^\gamma \widehat{\Phi}\left(\frac{|x_k|}{|1 + \|x(n)\|_{F\widehat{\Phi}}|}\right),\end{aligned}$$

and

$$\widehat{\Phi}(|\alpha||x_k(n) - x_k|) \approx |\alpha|^\gamma |x_k(n) - x_k|^\gamma \approx |\alpha|^\gamma \widehat{\Phi}\big(|x_k(n) - x_k|\big).$$

We know that $|\alpha(n) - \alpha|^\gamma \to 0$ and $\|x(n) - x\|_{F\widehat{\Phi}} \to 0$ as $n \to \infty$, consequently

$$\begin{aligned}\|(\alpha(n) - \alpha)(x(n) - x)\|_{F\widehat{\Phi}} &\approx |\alpha(n) - \alpha|^\gamma \|(x(n) - x)\|_{F\widehat{\Phi}} \to 0, \\ \|(\alpha(n) - \alpha)x\|_{F\widehat{\Phi}} &\approx |\alpha(n) - \alpha|^\gamma |1 + \|x\|_{F\widehat{\Phi}}|^\gamma \left\| \frac{x}{|1 + \|x\|_{F\widehat{\Phi}}|} \right\|_{F\widehat{\Phi}} \to 0, \\ \|\alpha(x(n) - x)\|_{F\widehat{\Phi}} &\approx |\alpha|^\gamma \|(x(n) - x)\|_{F\widehat{\Phi}} \to 0.\end{aligned}$$

We conclude that $\|\alpha(n)x(n) - \alpha x\|_{F\widehat{\Phi}} \to 0$ as $n \to \infty$.

Thus, $\|\cdot\|_{F\widehat{\Phi}}$ is an F-norm. ∎

At this stage, we have established that $(\ell^{\widehat{\Phi}}, \|\cdot\|_{F\widehat{\Phi}})$ forms an F-normed space, meaning the function $\|\cdot\|_{F\widehat{\Phi}}$ satisfies the fundamental properties of an F-norm. Further exploration could focus on its completeness.

4. The Completeness of $\big(\ell^{\widehat{\Phi}}, \|\cdot\|_{F\widehat{\Phi}}\big)$.

An important aspect of an F-normed space is its completeness. The following theorem states that $\big(\ell^{\widehat{\Phi}}, \|\cdot\|_{F\widehat{\Phi}}\big)$ is a complete space.

Theorem 4.1. *The space* $\big(\ell^{\widehat{\Phi}}, \|\cdot\|_{F\widehat{\Phi}}\big)$ *is complete.*

Proof. In $\ell^{\widehat{\Phi}}$, let $\big(x^{(n)}\big)$ be any Cauchy sequence with respect to $\|\cdot\|_{F\widehat{\Phi}}$, where

$$x^{(n)} = \left(x_k^{(n)}\right) = \left(x_1^{(n)}, x_2^{(n)}, \dots\right).$$

According to the definition of a Cauchy sequence, for every $\varepsilon > 0$, there exists an $n_0 \in \mathbb{N}$ such that for every $n, m > n_0$, it holds that

$$\sum_k \widehat{\Phi}\left(\left|x_k^{(n)} - x_k^{(m)}\right|\right) = \left\|x^{(n)} - x^{(m)}\right\|_{F\widehat{\Phi}} < \varepsilon.$$

As a result, for every $k \in \mathbb{N}$, we have $\left|x_k^{(n)} - x_k^{(m)}\right| \leq \Phi^{-1}(\varepsilon) = \bar{\varepsilon} > 0$. This shows that the sequence $\left(x_k^{(n)}\right)$ is Cauchy in $\mathbb{R}$. Since $\mathbb{R}$ is complete, then the sequence $\left(x_k^{(n)}\right)$ converges. That is, for each $k \in \mathbb{N}$, as $n \to \infty$, we have

$$x_k^{(n)} \to x_k.$$

Now, define $x = (x_k)$. It remains to show that $x \in \ell^{\widehat{\Phi}}$. Consider a sequence $\left(y^{(t)}\right)$, which is a subsequence of $\left(x^{(n)}\right)$, defined by

$$y^{(t)} := x^{(n_t)},$$

such that $\left|x_k^{(n_t)} - x_k\right| < \widehat{\Phi}^{-1}\left(\frac{1}{2^t}\frac{1}{2^k}\right)$. As a consequence, we have $\left\|x^{(n_t)} - x\right\|_{F\widehat{\Phi}} < \frac{C_1}{2^t}$, for some constant $C_1 > 0$. Therefore,

$$\left\|y^{(t)} - y^{(t+1)}\right\|_{F\widehat{\Phi}} \leq \left\|y^{(t)} - x\right\|_{F\widehat{\Phi}} + \left\|y^{(t+1)} - x\right\|_{F\widehat{\Phi}} \leq \frac{C_2}{2^t},$$

where $C_2 > 0$. We also define the sequence $\left(z^{(t)}\right)$ as

$$z_k^{(t)} := y_k^{(1)} + \sum_{j=1}^{t}\left|y_k^{(j+1)} - y_k^{(j)}\right|,$$

for each $k \in \mathbb{N}$. It follows that

$$\left\|z^{(t)}\right\|_{F\widehat{\Phi}} \leq \left\|y^{(1)}\right\|_{F\widehat{\Phi}} + \sum_{j=1}^{t}\left\|y^{(j+1)} - y^{(j)}\right\|_{F\widehat{\Phi}} \leq \left\|y^{(1)}\right\|_{F\widehat{\Phi}} + C_3\left(1 - \frac{1}{2^t}\right) < \infty.$$

Taking the limit as $t \to \infty$, we have

$$\left\|z^{(t)}\right\|_{F\widehat{\Phi}} \leq \|z\|_{F\widehat{\Phi}} \leq \left\|y^{(1)}\right\|_{F\widehat{\Phi}} + \sum_{j=1}^{\infty}\left\|y^{(j+1)} - y^{(j)}\right\|_{F\widehat{\Phi}} \leq \left\|y^{(1)}\right\|_{F\widehat{\Phi}} + C_3 < \infty,$$

which shows that $z^{(t)}$ and z belong to $\ell^{\widehat{\Phi}}$. Next, observe that

$$\begin{aligned}
\|x\|_{F\widehat{\Phi}} &= \|(x-z)+z\|_{F\widehat{\Phi}} \\
&\leq \|x-z\|_{F\widehat{\Phi}} + \|z\|_{F\widehat{\Phi}} \\
&\leq \left\|x - y^{(1)}\right\|_{F\widehat{\Phi}} + \left(\sum_{j=1}^{\infty}\left\|y^{(j+1)} - y^{(j)}\right\|_{F\widehat{\Phi}}\right) + \|z\|_{F\widehat{\Phi}} \\
&< \infty.
\end{aligned}$$

Hence, it follows that $x \in \ell^{\widehat{\Phi}}$.

We now demonstrate that the sequence $\left(x^{(n)}\right)$ also converges to x. Take $n < n_t$, then we observe

$$\left\|x^{(n)} - x\right\|_{F\widehat{\Phi}} = \left\|(x^{(n)} - y^{(t)}) + (y^{(t)} - x)\right\|_{F\widehat{\Phi}} \leq \left\|x^{(n)} - x^{(n_t)}\right\|_{F\widehat{\Phi}} + \left\|y^{(t)} - x\right\|_{F\widehat{\Phi}}.$$

Because $\left(x^{(n)}\right)$ forms a Cauchy sequence and $\left(y^{(t)}\right)$ converges to x, then it follows that as $n \to \infty$,

$$\left\|x^{(n)} - x^{(n_t)}\right\|_{F\widehat{\Phi}} \to 0 \quad \text{and} \quad \left\|y^{(t)} - x\right\|_{F\widehat{\Phi}} \to 0.$$

Consequently, we conclude that

$$\left\| x^{(n)} - x \right\|_{F\widehat{\Phi}} \to 0,$$

which shows that $\left(x^{(n)}\right)$ indeed converges to x. Finish. ∎

The above theorem provides a precise statement and proof that $(\ell^{\widehat{\Phi}}, \|\cdot\|_{F\widehat{\Phi}})$ is an F-space under appropriate conditions on $\widehat{\Phi}$.

We will present several examples of $\widehat{\Phi}$ as M-Young functions to construct $(\ell^{\widehat{\Phi}}, \|\cdot\|_{F\widehat{\Phi}})$ as F-normed spaces.

(1) Suppose $0 < s < 1$ and define $\widehat{\Phi}_*(t) := t^s$ for every $t \geq 0$. We obtain

$$\|x\|_{F\ell^s} = \|x\|_{F\widehat{\Phi}_*} = \sum_k |x_k|^s,$$

for every $x \in \ell^s$. The formula $\|\cdot\|_{F\ell^s}$ can be viewed as a usual F-norm. Next, we also have $\|\alpha x\|_{F\ell^s} = |\alpha|^s \|x\|_{F\ell^s}$ for every scalar α and $x \in \ell^s$. This means that $\|\cdot\|_{F\ell^s}$ is homogeneous of degree s, which makes easy to prove property (4) for $\|\cdot\|_{F\ell^s}$.

(2) Define $\widehat{\Phi}_1(t) := \ln(1+t)$ for every $t \geq 0$. We obtain

$$\|x\|_{F\widehat{\Phi}_1} = \sum_k \ln(1 + |x_k|),$$

and $(\ell^{\widehat{\Phi}_1}, \|\cdot\|_{F\widehat{\Phi}_1})$ forms an F-normed space. Note that for $0 \leq t < 1$, we have the Taylor series

$$\ln(1+t) = \sum_{j=1}^{\infty} \frac{(-1)^{j+1}}{j} t^j,$$

which gives the inequality

$$\frac{1}{2} t \leq \ln(1+t) \leq t.$$

As a result, by considering the normed space $(\ell^1, \|\cdot\|_{\ell^1})$, we define

$$\mathbb{A} := \{x \mid x \in \ell^1 \quad \text{such that} \quad \|x\|_{\ell^1} < 1\}.$$

We can form

$$\frac{1}{2}\|x\|_{\ell^1} \leq \|x\|_{F\widehat{\Phi}_1} \leq \|x\|_{\ell^1}$$

for every $x \in \mathbb{A}$. This means there is an equivalence between the F-norm $\|\cdot\|_{F\widehat{\Phi}_1}$ and $\|\cdot\|_{\ell^1}$ on the normalized ℓ^1 space (which contains vectors with a "length" less than or equal 1). In the normalized case, we have $\ell^1 = \ell^{\widehat{\Phi}_1}$. The equivalence mentioned above helps in proving property (4) for $\|\cdot\|_{F\widehat{\Phi}_1}$.

From the examples presented, the F-normed spaces are also complete spaces. This is quite straightforward, and readers are encouraged to verify it.

5. Concluding Remarks

Based on this study, it is possible to continue by examining the inclusion properties in $(\ell^{\widehat{\Phi}}, \|\cdot\|_{F\widehat{\Phi}})$, if we consider suitable different functions $\widehat{\Phi}$. Furthermore, it is plausible that for arbitrary $\widehat{\Phi}$, a quasi-norm can be defined, and its properties could be further explored.

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