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On fixed point theorems in modular spaces characterized by C^*-class functions

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Abstract Generally, finding a solution to a theoretical mathematical modelling problem is equivalent to finding a fixed point for a suitable operator. Accordingly, fixed point theory is therefore very important and crucial in many areas, such as mathematics, sciences, and engineering. A very popular and important fixed point theory is that formulated by Stefan Banach in 1922. The theory is related to a complete normed space and known as the Banach fixed point theory.
Recently there have been numerous generalizations of the Banach fixed point theory. One of them is a fixed point theory in modular spaces. In this paper, we will formulate some fixed point theorems in modular spaces by using C^*-class functions. The obtained results generalize and improve some results in [21].

MSC: 46A40; 46B40; 47H10
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1. Introduction

In a wide range of mathematical modelling problems, finding a solution of real or theoretical problem is equivalent to finding a fixed point for a suitable operator or mapping. Accordingly, a fixed point theory is therefore a very important and crucial in many areas, such as mathematics, sciences, economics, and engineering. A very popular and important fixed point theory is those formulated by Stefan Banach in 1922 [4]. The theory is related to a complete normed space and known as the Banach fixed point theory [14, 19]. Due to its importance, many researchers then extended and generalized those theory via some various ways, such as by replacing normed spaces by modular spaces (See for e.g. [1, 2, 7, 8, 10–13, 15, 20–22]).

The theory of modular spaces was firstly initiated by H. Nakano in 1950 [17]. Initially, Nakano defined a modular function on an order vector space, that is a vector space equipped with an order relation such that the order and the vector space structure are compatible. Later on, the Nakano's definition was generalized by Orlicz and Musielak in

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1959 by omitting the order structure in the order vector space in the Nakano's version [16, 18]. Based on the definition from Orlicz and Musielak, a modular is therefore a general case of a norm. Modular metric spaces are a natural generalization of classical modulars over linear spaces like Lebesgue spaces, Orlicz spaces, Musielak-Orlicz spaces, and many others [5, 6, 9].

Over time, many results in fixed point theory generalize those of Banach for modular spaces (See for e.g. [1, 2, 7, 8, 10–13, 15, 20–22]). In this paper, we formulate some fixed point theorems in modular spaces by using C^*-type functions. The results generalize the theory in [21].

2. Some Basic Notion and and Preliminaries

In this section, we recall some essential definitions and fundamental results [21].

As usual, $\mathbb{N}$ and $\mathbb{R}$ denote the set of all positive integers and real numbers system, respectively. The extended real numbers system will be denoted by $\mathbb{R}^*$.

Let X be a linear space over $\mathbb{R}$. A non-negative function $\rho : X \to \mathbb{R}^*$ is called a *modular* if for every $f, g \in X$ the following conditions hold.

(i) $\rho(f) = 0$ iff $f = 0$.
(ii) $\rho(-f) = \rho(f)$.
(iii) $\rho(\alpha f + \beta g) \leq \rho(f) + \rho(g)$ for every $\alpha, \beta \geq 0$ such that $\alpha + \beta = 1$.

If we change the axiom (iii) by

(iii') $\rho(\alpha f + \beta g) \leq \alpha\rho(f) + \alpha\rho(g)$ for every $\alpha, \beta \geq 0$ such that $\alpha + \beta = 1$,

then we say that the modular ρ is a convex modular. A linear space X equipped with a modular ρ, written by (X, ρ), is called a modular space. We shall also denote a modular space by the single character X, when the modular ρ is explicitly understood.

By considering the definition of the modular, then we can easily prove the following theorems.

Theorem 2.1. *Let (X, ρ) be a modular space.*

(i) If $\alpha, \beta \in \mathbb{R}$, $0 \leq \alpha \leq \beta$ then $\rho(\alpha f) \leq \rho(\beta f)$ for every $f \in X$.
(ii) If $\rho(f) < \epsilon$ for every $\epsilon > 0$ then $f = 0$.

Theorem 2.2. *Let (X, ρ) be a modular space. If $f_1,\ f_2,\ f_3, \ldots, f_n \in X$, and $\alpha_1,\ \alpha_2,\ \alpha_3, \ldots, \alpha_n$ are non-negative real numbers such that $\sum_{i=1}^n \alpha_i = 1$, then $\rho(\sum_{i=1}^n \alpha_i f_i) \leq \sum_{i=1}^n \rho(f_i)$.*

Theorem 2.3. *Let (X, ρ) be a modular space. If the modular ρ is convex, then for any $f_1,\ f_2,\ f_3, \ldots, f_n \in X$ and any non-negative real numbers $\alpha_1,\ \alpha_2,\ \alpha_3, \ldots, \alpha_n$ satisfying $\sum_{i=1}^n \alpha_i = 1$ we have $\rho(\sum_{i=1}^n \alpha_i f_i) = \sum_{i=1}^n \alpha_i \rho(f_i)$.*

Definition 2.4. The modular function ρ on X_ρ is said to satisfy the Δ_2-condition if there exists $K > 0$ such that $\rho(2x) \leq K\rho(x)$ for any $x \in X_\rho$.

Throughout this paper, we assume that the modular ρ is always convex and satisfying the Δ_2-condition, unless otherwise stated.

Let (X, ρ) be a modular space. We can show that the set

$$X_\rho = \{f \in X : \ \rho(f) < \infty\} \tag{2.1}$$

is a linear space, modulared by ρ. It can be also verified that $\rho(f) < \infty$ for every $f \in X_\rho$. In this paper, we always mean that the modular space X_ρ is as given in (2.1).

Let (X, ρ) be a modular space. A sequence $\left(f^{(n)}\right)$ in X_ρ is said to be *ρ-convergent* (*modular convergent*) to $f \in X_\rho$ if for every real number $\epsilon > 0$ there exists a positive integer N such that for every $n \geq N$, we have:

$$\rho\left(f^{(n)} - f\right) < \epsilon.$$

In this case, f is called a *modular limit* (*ρ-limit*) of $\left(f^{(n)}\right)$, and we write

$$\rho\text{-}\lim_{n\to\infty} f^{(n)} = f.$$

If the sequence $\left(f^{(n)}\right)$ in X_ρ is ρ-convergent, then its ρ-limit is unique. A sequence $\left(f^{(n)}\right)$ in X_ρ is called a *ρ-Cauchy* (*modular Cauchy*) sequence if for every real number $\epsilon > 0$ there exists a positive integer N such that for every $m, n \geq N$, we have:

$$\rho\left(f^{(n)} - f^{(m)}\right) < \epsilon.$$

It is easy to check that in every modular space, every ρ-convergent sequence is ρ-Cauchy sequence. The modular space X_ρ is said to be ρ-complete if every ρ-Cauchy sequence in X_ρ is ρ-convergent.

Definition 2.5. Any set $E \subset X_\rho$ is said to be modular close (ρ-closed) if for any sequence (f_n) in E which is ρ-convergent to $f \in X_\rho$ implies $f \in E$.

Definition 2.6. Any set $B \subset X_\rho$ is said to be modular bounded (ρ-bounded) if there exists an $M > 0$ such that $\rho(f - g) < M$ for every $f, g \in X_\rho$. It is equivalent to say that $B \subset X_\rho$ is modular bounded iff $\sup\{\rho(f-g) : \ f,\ g \in B\} < \infty$.

In 2014, A.H. Ansari [3] proved the existence of C-class functions that cover a large class of contractive conditions. We revise the definition of A.H. Ansari to get the more general one, as given in the following definition.

Definition 2.7. A function $f : [0, \infty) \times [0, \infty) \to \mathbb{R}$ is said to be *C^*-type* , if the following conditions hold.

(i) If (s_n) and (t_n) are any convergent sequences in $[0, \infty)$, then $\lim f(s_n, t_n) = f(\lim s_n, \lim t_n)$.
(ii) $f(s, t) \leq s$ for any $s, t \in [0, \infty)$.
(iii) If $f(s, t) = s$, then $s = 0$ or $t = 0$.

It is clear that condition (ii) in Definition 2.7 implies $f(0, 0) = 0$. We can also see that any C-class function is of C^*-type.

Example 2.8. A function $f(s, t) = s - t,\ s, t \in [0, \infty)$, is C^*-type. However, a function $g(s, t) = s + t,\ s, t \in [0, \infty)$, is not C^*-type.

3. Some Fixed Point Theorems Characterized by C^*-Type Functions

The collection of all continuous functions $\varphi : [0, \infty) \to [0, \infty)$ such that $\varphi(t) > 0$ for every $t > 0$ and $\varphi(0) \geq 0$ will be denoted by $C^+[0, \infty)$. For any operators $S, T : X \to X$, a product ST is meant as a composition of T and S, i.e.

$$(ST)(f) = S(T(f)),$$

for any $f \in X$. Further, we will present some results related to fixed points, which are formulated using a function of C^*-type. The first one is formulated in the following theorem.

Theorem 3.1. *Let (X_ρ, ρ) be a ρ-complete modular space and $B \subset X_\rho$ be a ρ-closed and ρ-bounded set. If $S, T : B \to B$ are operators satisfying $ST = TS$, F is a function of C^*-type, and $\varphi \in C^+[0, \infty)$, such that*

$$\rho\left(T(f) - T(g)\right) \leq F\left(\rho(S(f) - S(g)), \varphi(\rho(S(f) - S(g)))\right) \tag{3.1}$$

for every $f, g \in B$, then S and T have a unique common fixed point.

Proof. Let $f_0 \in B$. For any integer $n \geq 0$, we define

$$S(f_{n+1}) = T(f_n).$$

Let $\alpha_0 = \rho(S(f_0))$ and $\alpha_{n+1} = \rho(S(f_{n+1}) - S(f_n))$ for every $n \in \mathbb{N} \cup \{0\}$. Since, F is C^*-type, then for any $n \in \mathbb{N}$, we have

$$\begin{aligned} \alpha_{n+1} &= \rho(T(f_n) - T(f_{n-1})) \\ &\leq F(\rho(S(f_n) - S(f_{n-1})), \varphi(\rho(S(f_n) - S(f_{n-1})))) \\ &\leq \rho(S(f_n) - S(f_{n-1})) = \alpha_n \end{aligned}$$

This implies that the sequence (α_n) converges to some $r \geq 0$. Hence,

$$r = F(r, \varphi(r)).$$

Following the assumption of F, then $r = 0$ or $\varphi(r) = 0$, which yields $r = 0$ or

$$\lim_{n\to\infty} \rho(S(f_{n+1}) - S(f_n)) = 0. \tag{3.2}$$

The next step, we will show that $(T(f_n))$ is a ρ-Cauchy sequence. By assuming the contrary and by noticing equation (3.2), then there exists $\varepsilon > 0$ such that we can find two sequences (m_k) and (n_k) of positive integers satisfying $n_k > m_k \geq k$ such that

$$\rho\left(T(f_{n_k}) - T(f_{m_k})\right) \geq \varepsilon \quad \text{and} \quad \rho\left(2(T(f_{n_k-1}) - T(f_{m_k}))\right) < \varepsilon. \tag{3.3}$$

Following (3.3) and Theorem 2.2, then we have

$$\begin{aligned} \varepsilon &\leq \rho\left(T(f_{n_k}) - T(f_{m_k})\right) \\ &\leq \rho\left(2(T(f_{n_k}) - T(f_{n_k-1}))\right) + \rho\left(2(T(f_{n_k-1}) - T(f_{m_k}))\right) \\ &< \rho\left(2(T(f_{n_k}) - T(f_{n_k-1}))\right) + \varepsilon. \end{aligned} \tag{3.4}$$

So, by taking the limit as $k \to \infty$ for each side of (3.4), we obtain that

$$\lim_{k\to\infty} \rho\left(T(f_{n_k}) - T(f_{m_k})\right) = \varepsilon. \tag{3.5}$$

Take $f = f_{n_k}$ and $g = f_{m_k-1}$, then from (3.1) we get

$$\begin{aligned} \rho(T(f_{n_k}) - T(f_{m_k})) &\leq F(\rho(S(f_{n_k}) - S(f_{m_k})), \varphi\left(\rho(S(f_{n_k}) - S(f_{m_k}))\right)) \\ &\leq F(\rho(T(f_{n_k-1}) - T(f_{m_k-1})), \varphi\left(\rho(T(f_{n_k-1}) - T(f_{m_k-1}))\right)). \end{aligned} \tag{3.6}$$

Hence, by taking the limit as $k \to \infty$ on each side of (3.6) and following (3.5) and Definition 2.7, we get

$$\varepsilon \leq F(\varepsilon, \varphi(\varepsilon)).$$

This contradicts to the assumption that F is a C^*-type function. Thus, $(T(f_n))$ is a ρ-Cauchy sequence. Following the definition of $S(f_n)$, then $(S(f_n))$ is a ρ-Cauchy sequence as well. Furthermore, following the ρ-completeness of X_ρ, then $(S(f_n))$ and $(T(f_n))$ are ρ-convergent in X_ρ, and by considering the construction of (f_n), then we have

$$\lim_{n\to\infty} T(f_n) = \lim_{n\to\infty} S(f_n) = g$$

for some $g \in X_\rho$. Since, B is ρ-closed, then $g \in B$. As the consequence, there exists an $f \in B$ such that

$$T(f) = g = S(f). \tag{3.7}$$

Since $ST = TS$, then (3.7) implies that $T(T(f)) = S(S(f))$. Furthermore,

$$\begin{aligned}\rho\left(T(f) - T^2(f)\right) &\leq F(\rho(S(f) - S(S(f))), \varphi(\rho(S(f) - S(S(f)))))\\ &= F\left(\rho(T(f) - T^2(f)), \varphi(\rho(T(f) - T^2(f)))\right).\end{aligned}$$

So, $\rho\left(T(f) - T^2(f)\right) = 0$ or $\varphi\left(\rho(T(f) - T^2(f))\right) = 0$, which yields $\rho\left(T(f) - T^2(f)\right) = 0$. This implies

$$T(f) = T(T(f)) = S(T(f))$$

i.e. $T(f) = g \in B$ is a common fixed point of T and S. The uniqueness of the fixed point follows from the convexity of the modular ρ. ∎

Theorem 3.2. *Let (X_ρ, ρ) be a ρ-complete modular space and $B \subset X_\rho$ be a ρ-closed and ρ-bounded set. If operators $S, T : B \to B$ satisfy $ST = TS$, $\varphi \in C^+[0, \infty)$, and F is a function of C^*-type, such that*

$$\rho(2(T(f) - T(g))) \leq 2F(\rho(S(f) - S(g)), \varphi(\rho(S(f) - S(g)))) \tag{3.8}$$

for every $f, g \in B$, then S and T have a unique common fixed point.

Proof. Let $f_0 \in B$. For any integer $n \geq 0$, we define

$$S(f_{n+1}) = T(f_n)$$

Let $\alpha_0 = \rho(S(f_0))$ and $\alpha_{n+1} = \rho(S(f_{n+1}) - S(f_n))$ for every $n \in \mathbb{N} \cup \{0\}$. Since ρ is convex, then for any $n \in \mathbb{N}$, we have

$$\alpha_{n+1} \leq \frac{1}{2}\rho(2(T(f_n) - T(f_{n-1}))).$$

So, by following the hypothesis, we obtain

$$\begin{aligned}\alpha_{n+1} &\leq F(\rho(S(f_n) - S(f_{n-1})), \varphi(\rho(S(f_n) - S(f_{n-1}))))\\ &\leq \rho(S(f_n) - S(f_{n-1})) = \rho(T(f_{n-1}) - T(f_{n-2})) = \alpha_n.\end{aligned} \tag{3.9}$$

This implies $\alpha_n \to r$ for some $r \geq 0$. So, by letting $n \longrightarrow \infty$ for (3.9), we obtain

$$r = F(r, \varphi(r)).$$

Since F is C^*-type, then $r = 0$ or $\varphi(r) = 0$, which yields $r = 0$ or

$$\lim_{n\to\infty} \rho(S(f_{n+1}) - S(f_n)) = 0 \tag{3.10}$$

Next, we will prove that $(T(f_n))$ is a ρ-Cauchy sequence. Assume on the contrary. By noticing (3.10), then there exists $\varepsilon > 0$ such that we can find two sequences (m_k) and (n_k) of positive integers satisfying $n_k > m_k \geq k$ such that the following inequalities hold.

$$\rho\left(T(f_{n_k}) - T(f_{m_k})\right) \geq \varepsilon \quad \text{and} \quad \rho\left(2(T(f_{n_k-1}) - T(f_{m_k}))\right) < \varepsilon. \tag{3.11}$$

Following (3.11) and Theorem 2.2, then we have

$$\begin{aligned}\varepsilon &\leq \rho\left(T(f_{n_k}) - T(f_{m_k})\right)\\ &= \rho\left(T(f_{n_k}) - T(f_{n_k-1}) + T(f_{n_k-1}) - T(f_{m_k})\right)\\ &\leq \rho\left(2(T(f_{n_k}) - T(f_{n_k-1}))\right) + \rho\left(2(T(f_{n_k-1}) - T(f_{m_k}))\right)\\ &< \varepsilon + M\rho\left(T(f_{n_k}) - T(f_{n_k-1})\right),\end{aligned}$$

for some $M > 0$. So, by taking the limit as $k \to \infty$, we obtain that

$$\lim_{k\to\infty} \rho\left(T(f_{n_k}) - T(f_{m_k})\right) = \varepsilon$$

Further, take $f = f_{n_k}$ and $g = f_{m_k-1}$ in (3.8), then we have

$$\begin{aligned}\rho(T(f_{n_k}) - T(f_{m_k})) &\leq \frac{1}{2}\rho(2(T(f_{n_k}) - T(f_{m_k})))\\ &\leq F(\rho(S(f_{n_k}) - S(f_{m_k})), \varphi\left(\rho(S(f_{n_k}) - S(f_{m_k}))\right))\\ &= F(\rho(T(f_{n_k-1}) - T(f_{m_k-1})), \varphi\left(\rho(T(f_{n_k-1}) - T(f_{m_k-1}))\right))\end{aligned} \tag{3.12}$$

By taking the limit as $k \to \infty$ on each side of (3.12) and by using (3.10) and Definition 2.7, we get

$$\varepsilon \leq F(\varepsilon, \varphi(\varepsilon)).$$

This contradicts to the assumption of F. Thus, $(T(f_n))$ is a ρ-Cauchy sequence. As a consequence, $(S(f_n))$ is also a ρ-Cauchy sequence. Hence, because of the ρ-completeness of X_ρ, then $(S(f_n))$ and $(T(f_n))$ are ρ-convergent in X_ρ. And by considering the construction of (f_n), then we have

$$\lim_{n\to\infty} T(f_n) = \lim_{n\to\infty} S(f_n) = g$$

for some $g \in X_\rho$. Since, B is ρ-closed, then $g \in B$. Further, there exists an $f \in B$ such that

$$T(f) = g = S(f).$$

And since $ST = TS$, then $T(T(f)) = S(S(f))$. Furthermore,

$$\begin{aligned}\rho\left(T(f) - T^2(f)\right) &\leq \frac{1}{2}\rho(2(T(f) - T^2(f)))\\ &\leq F(\rho(S(f) - S(S(f))), \varphi(\rho(S(f) - S(S(f))))\\ &= F\left(\rho(T(f) - T^2(f)), \varphi(\rho(T(f) - T^2(f)))\right).\end{aligned}$$

Therefore, by the hypothesis,

$$\rho\left(T(f) - T^2(f)\right) = 0 \quad \text{or} \quad \varphi\left(\rho(T(f) - T^2(f))\right) = 0,$$

which yields $\rho\left(T(f) - T^2(f)\right) = 0$. So, we have

$$T(f) = T(T(f)) = S(T(f)),$$

i.e. $T(f) = g \in B$ is a common fixed point of T and S. The uniqueness of $T(f)$ follows from the convexity of the modular ρ. ■

Following the Theorem 3.1 and Theorem 3.2, we have the following corollaries.

Corollary 3.3. *[21] Let (X_ρ, ρ) be a ρ-complete modulared space, where the modular ρ is convex, $B \subset X_\rho$ ρ-closed and ρ-bounded set. If the operators $S, T : B \to B$ satisfying $ST = TS$ and*

$$\rho(T(f) - T(g)) \leq k\rho(S(f) - S(g))$$

for every $f, g \in B$ and for some $k \in (0, 1)$, then S and T have a unique common fixed point.

Proof. By taking $F(s, t) = ks, k \in (0, 1)$ in Theorem 3.1, then the assertion follows. ∎

Corollary 3.4. *[21] Let (X_ρ, ρ) be a ρ-complete modulared space, where the modular ρ is convex, $B \subset X_\rho$ ρ-closed and ρ-bounded set . If operators $S, T : B \to B$ satisfying $ST = TS$ and*

$$\rho(2(T(f) - T(g))) \leq k\rho(S(f) - S(g))$$

for every $f, g \in B$ and for some $k \in (0, 2)$, then S and T have a unique common fixed point.

Proof. Take $F(s, t) = ks, k \in (0, 1)$ in Theorem 3.2, then the assertion follows. ∎

Corollary 3.5. *[21] Let (X_ρ, ρ) be a ρ-complete modulared space, where the modular ρ is convex, $B \subset X_\rho$ ρ-closed and ρ-bounded set, and $\phi \in C^+[0, \infty)$. If operators $S, T : B \to B$ satisfying $ST = TS$ and*

$$\rho(T(f) - T(g)) \leq \phi(\rho(S(f) - S(g)))$$

for every $f, g \in B$, then S and T have a unique common fixed point..

Proof. Apply Theorem 3.1 by choosing $F(s, t) = \phi(s)$, where $\phi \in C^+[0, \infty)$, then the assertion follows. ∎

Corollary 3.6. *Let (X_ρ, ρ) be a ρ-complete modulared space, where the modular ρ is convex, $B \subset X_\rho$ ρ-closed and ρ-bounded set, and $\phi \in C^+[0, \infty)$. If operators $S, T : B \to B$ satisfying $ST = TS$ and*

$$\rho(2(T(f) - T(g))) \leq 2\phi(\rho(S(f) - S(g)))$$

for every $f, g \in B$, then S and T have a unique common fixed point.

Proof. Take $F(s, t) = \phi(s)$, where $\phi \in C^+[0, \infty)$ and apply Theorem 3.2, then the assertion follows. ∎

4. Concluding Remarks

Some fixed point theorems in modular spaces have been able to be formulated by using C^*-type functions. The results generalize the theorems in [21].

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