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Simplified computation of useful functions in linear canonical transform

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Abstract The linear canonical transform is an extension of the usual Fourier transform because the Fourier transform is a special form of the linear canonical transform. It is also a valuable tool in signal analysis. Many essential properties of the Fourier transform can be transferred to the linear canonical Fourier domain with some changes. In this research paper, we first introduce the interesting connection between the linear canonical transform and the Fourier transform. It is shown that the relation can be developed to efficiently evaluate the Gaussian function in the linear canonical transform domain. Some examples of the Gaussian function in the linear canonical domain are also presented to illustrate the result.

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1. Introduction

As is known to all, the linear canonical transform (LCT) is the linear integral transformation that has a quadratic-phase kernel. That is why the linear canonical transform is usually called as the quadratic-phase transformation. In the recent ten years, a detailed discussion of the main properties of the linear canonical transform, such as translation, dilation, uncertainty principles, and the convolution theorem, has been investigated in [1–3, 5, 13].

On the flip side, many famous transformations like the classical Fourier transform and fractional Fourier transform, Laplace transform, chirp multiplication, and Fresnel transformation are particular cases of the LCT. Compared with the fractional Fourier transform with three degrees of freedom and the classical Fourier transform without a parameter, the LCT is more flexible due to its extra parameter. Some important applications of this extended transformation in the fields of optics, filter design, signal analysis, time-frequency analysis, encryption, modulation, and multiplexing in communications have been demonstrated (see [6–12, 14] and elsewhere).

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In this research paper, we consider the direct connection between the linear canonical transform and the Fourier transform. It is shown that the relation can be developed to efficiently evaluate the Gaussian function in the linear canonical transform domain. Some examples of the Gaussian function in the linear canonical domain are also presented to illustrate the result.

2. Definition of Linear Canonical Transform

In this section, we introduce the definition of the linear canonical transform (LCT) and its inverse.

Definition 2.1. The definition of the LCT with matrix parameter $B = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathbb{R}^{2\times 2}$ for which $\det(B) = ad - bc = 1$ and $f(t) \in L^2(\mathbb{R})$ is expressed as

$$L_B\{f(t)\}(\omega) = \begin{cases} \int_{-\infty}^{\infty} f(t) K_B(\omega, t) dt & b \neq 0, \\ \sqrt{d} e^{\frac{i}{2} cd\omega^2} f(d\omega) & b = 0. \end{cases} \tag{2.1}$$

where $K_B(\omega, t)$ is the LCT kernel given as

$$K_B(\omega, t) = \frac{1}{\sqrt{2\pi b}} e^{\frac{i}{2}\left(\frac{a}{b}t^2 - \frac{2}{b}t\omega + \frac{d}{b}\omega^2 - \frac{\pi}{2}\right)}.$$

Without loss of generality, in relation (2.1) we always consider $b \neq 0$ throughout the article. Further, the signal $f(t)$ can be recovered using the inversion formula for the LCT as follows.

Theorem 2.2. *For every $f(t) \in L^1(\mathbb{R})$ with $L_B\{f(t)\} \in L^1(\mathbb{R})$, the inverse formula of the LCT is described through*

$$\begin{aligned} f(t) &= \int_{-\infty}^{\infty} L_B\{f(t)\}(\omega) \overline{K_B(\omega, t)} d\omega \\ &= \int_{-\infty}^{\infty} L_B\{f(t)\}(\omega) \frac{1}{\sqrt{2\pi b}} e^{-\frac{i}{2}\left(\frac{a}{b}t^2 - \frac{2}{b}t\omega + \frac{d}{b}\omega^2 - \frac{\pi}{2}\right)} d\omega \end{aligned}$$

where $\overline{K_B(\omega, t)}$ represents the conjugation of $K_B(\omega, t)$.

3. Linear Canonical Transform and its Relation to Fourier Transform

In this segment, we mainly show that we can evaluate the Gaussian signal in the linear canonical transform domain by expounding on the direct relationship between the Fourier transform and the linear canonical transform. For this, we provide the following definition (see [4]).

Definition 3.1. The Fourier transform of $f \in L^2(\mathbb{R})$, is the function defined by the integral form

$$\mathcal{F}\{f\}(\omega) = \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt. \tag{3.1}$$

By exploiting equation (2.1), the linear canonical transform definition can be reduced to the Fourier transform definition, that is,

$$\begin{aligned} L_B\{f(t)\}(\omega) &= \int_{-\infty}^{\infty} f(t)\frac{1}{\sqrt{2\pi b}} e^{\frac{i}{2}\left(\frac{a}{b}t^2-\frac{2}{b}t\omega+\frac{d}{b}\omega^2-\frac{\pi}{2}\right)}dt \\ &= \frac{1}{\sqrt{2\pi b}} e^{\frac{i}{2}\left(\frac{d}{b}\omega^2-\frac{\pi}{2}\right)}\int_{-\infty}^{\infty} f(t)e^{\frac{i}{2}\left(\frac{a}{b}t^2-\frac{2}{b}t\omega\right)}dt \\ &= \frac{1}{\sqrt{2\pi b}} e^{\frac{i}{2}\left(\frac{d}{b}\omega^2-\frac{\pi}{2}\right)}\mathcal{F}\{h\}\left(\frac{\omega}{b}\right). \end{aligned} \tag{3.2}$$

This is equivalent to

$$\mathcal{F}\{h\}\left(\frac{\omega}{b}\right) = \sqrt{2\pi b}e^{-\frac{i}{2}\left(\frac{d}{b}\omega^2-\frac{\pi}{2}\right)}L_B\{f(t)\}(\omega),$$

where $h(t) = f(t)e^{\frac{ia}{2b}t^2}$.

The above relation allows us to evaluate the Gaussian signal in the linear canonical Fourier domain. Given the Gaussian function in the following form:

$$f(t) = e^{-kt^2} \quad \text{with} \quad k > 0,$$

as shown in Figure 1.

From the Fourier transform in the Definition 3.1, it follows that

$$\begin{aligned} \mathcal{F}\{h\}\left(\frac{\omega}{b}\right) &= \mathcal{F}\left\{f(t)e^{\frac{ia}{2b}t^2}\right\}\left(\frac{\omega}{b}\right) \\ &= \mathcal{F}\left\{e^{-kt^2}e^{\frac{ia}{2b}t^2}\right\}\left(\frac{\omega}{b}\right) \\ &= \int_{-\infty}^{\infty} e^{it^2\frac{a}{2b}}e^{-kt^2}e^{-it\frac{\omega}{b}}dt \\ &= \int_{-\infty}^{\infty} e^{-\frac{1}{2b}(2bk-ia)t^2}e^{-it\frac{\omega}{b}}dt \\ &= \int_{-\infty}^{\infty} e^{-\frac{1}{2b}(2bk-ia)\left(t^2+\frac{it\frac{\omega}{b}}{\frac{1}{2b}(2bk-ia)}\right)}dt \\ &= \int_{-\infty}^{\infty} e^{-\frac{1}{2b}(2bk-ia)\left(\left(t+\frac{i\frac{\omega}{b}}{2\left(\frac{1}{2b}(2bk-ia)\right)}\right)^2-\left(\frac{i\frac{\omega}{b}}{2\left(\frac{1}{2b}(2bk-ia)\right)}\right)^2\right)}dt \\ &= \int_{-\infty}^{\infty} e^{-\frac{1}{2b}(2bk-ia)\left(\left(t+\frac{i\omega}{2bk-ia}\right)^2+\frac{\left(\frac{\omega}{b}\right)^2}{4\left(\frac{1}{2b}(2bk-ia)\right)^2}\right)}dt \\ &= e^{-\left(\frac{\left(\frac{\omega}{b}\right)^2}{\frac{2}{b}(2bk-ia)}\right)}\int_{-\infty}^{\infty} e^{-\left(\frac{2bk-ia}{2b}\right)\left(t+\frac{i\omega}{2bk-ia}\right)^2}dt \\ &= e^{-\frac{\omega^2}{2b(2bk-ia)}}\int_{-\infty}^{\infty} e^{-\left(\frac{2bk-ia}{2b}\right)\left(t+\frac{i\omega}{2bk-ia}\right)^2}dt. \end{aligned}$$

Now putting

$$\alpha = \frac{2bk-ia}{2b}, \quad y = t+\frac{i\omega}{2bk-ia}, \tag{3.3}$$

then, we get

$$\mathcal{F}\{h\}\left(\frac{\omega}{b}\right) = e^{-\frac{\omega^2}{2b(2bk-ia)}} \int_{-\infty}^{\infty} e^{-\alpha y^2}\,dy.$$

Hence,

$$\mathcal{F}\{h\}\left(\frac{\omega}{b}\right) = e^{-\frac{\omega^2}{2b(2bk-ia)}} \sqrt{\frac{\pi}{\alpha}}. \tag{3.4}$$

Substituting the first term of relation (3.3) into relation (3.4), one could easily obtain

$$\mathcal{F}\{h\}\left(\frac{\omega}{b}\right) = e^{-\frac{\omega^2}{2b(2bk-ia)}} \sqrt{\frac{2\pi b}{2bk-ia}}.$$

It follows from relation (3.2) that

$$\begin{aligned}
L_B\{f\}(\omega) &= \frac{1}{\sqrt{2\pi b}} e^{\frac{i}{2}\left(\frac{d}{b}\omega^2 - \frac{\pi}{2}\right)} \sqrt{\frac{2\pi b}{2bk-ia}} e^{-\frac{\omega^2}{2b(2bk-ia)}} \\
&= \sqrt{\frac{2\pi b}{(2\pi b)(2bk-ia)}} e^{\frac{i}{2}\left(\frac{d}{b}\omega^2 - \frac{\pi}{2}\right)} e^{-\frac{\omega^2}{2b(2bk-ia)}} \\
&= \frac{1}{\sqrt{2bk-ia}} e^{\frac{\omega^2}{2b}\left(id - \frac{1}{2bk-ia}\right) - i\frac{\pi}{4}} \\
&= \frac{1}{\sqrt{2bk-ia}} e^{\frac{\omega^2}{2b}\left(\frac{id(2bk-ia)-(ad-bc)}{2bk-ia}\right) - i\frac{\pi}{4}} \\
&= \frac{1}{\sqrt{2bk-ia}} e^{\frac{\omega^2}{2b}\left(\frac{2bdik+ad-ad+bc}{2bk-ia}\right) - i\frac{\pi}{4}} \\
&= \frac{1}{\sqrt{2bk-ia}} e^{\frac{\omega^2}{2}\left(\frac{c+2dik}{2bk-ia}\right) - i\frac{\pi}{4}}.
\end{aligned}$$

As a special case, when the matrix parameter $B = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ we immediately obtain

$$L_B\{f\}(\omega) = \frac{1}{\sqrt{2\pi i}} \sqrt{\frac{\pi}{k}} e^{-\frac{\omega^2}{4k}},$$

which shows the above equation is consistent with the Gaussian function in the Fourier transform domain. Figures 2(a) and 2(b) illustrate the Gaussian signal in the linear canonical Fourier domain under the different matrix parameters.

Further, let us evaluate the norm (modulus) of the Gaussian signal in the linear canonical transform domain. In fact, we have

$$\begin{aligned}
|L_B\{f\}(\omega)| &= \left| \frac{1}{\sqrt{2bk-ia}} e^{\frac{\omega^2}{2}\left(\frac{c+2dik}{2bk-ia}\right) - i\frac{\pi}{4}} \right| \\
&= \left| \frac{1}{\sqrt{2bk-ia}} \right| \left| e^{\frac{\omega^2}{2}\left(\frac{c+2dik}{2bk-ia}\right) - i\frac{\pi}{4}} \right|.
\end{aligned} \tag{3.5}$$

Now observe that

$$\begin{aligned}
\left|\frac{1}{\sqrt{2bk-ia}}\right| &= \left|\sqrt{\frac{1}{2bk-ia}}\right| \\
&= \left|\frac{\sqrt{2bk+ia}}{\sqrt{4b^2k^2+a^2}}\right| \\
&= \frac{1}{\sqrt{4b^2k^2+a^2}}\left|(2bk+ia)^{\frac{1}{2}}\right| \\
&= \frac{1}{\sqrt{4b^2k^2+a^2}}\left|(2bk+ia)\right|^{\frac{1}{2}} \\
&= \frac{1}{\sqrt{4b^2k^2+a^2}}(4b^2k^2+a^2)^{\frac{1}{4}} \\
&= (4b^2k^2+a^2)^{-\frac{1}{4}}.
\end{aligned}$$

On the other hand, we find that

$$\begin{aligned}
\left|e^{\frac{\omega^2}{2}\left(\frac{c+2dki}{2bk-ia}\right)}\right| &= \left|e^{\frac{\omega^2}{2}\left(\frac{(c+2dki)(2bk+ia)}{4b^2k^2+a^2}\right)}\right| \\
&= \left|e^{\frac{\omega^2}{2}\left(\frac{2cbk-2dka}{4b^2k^2+a^2}\right)+i\frac{\omega^2}{2}\left(\frac{ac+4k^2db}{4b^2k^2+a^2}\right)}\right| \\
&= \left|e^{\frac{\omega^2}{2}(\frac{2cbk-2dka}{4b^2k^2+a^2}}e^{\frac{i\omega^2}{2}\frac{ac+4k^2db}{4b^2k^2+a^2}}\right| \\
&= \left|e^{\frac{\omega^2}{2}\left(\frac{2cbk-2dka}{4b^2k^2+a^2}\right)}\right|\underbrace{\left|e^{\frac{i\omega^2}{2}\left(\frac{ac+4k^2db}{4b^2k^2+a^2}\right)}\right|}_{1} \\
&= e^{\frac{\omega^2}{2}\left(\frac{2cbk-2dka}{4b^2k^2+a^2}\right)}.
\end{aligned}$$

Therefore, we finally get

$$\begin{aligned}
|L_B\{f\}(\omega)| &= (4b^2k^2+a^2)^{-\frac{1}{4}}e^{\frac{\omega^2}{2}\left(\frac{2cbk-2dka}{4b^2k^2+a^2}\right)} \\
&= \frac{1}{(4b^2k^2+a^2)^{\frac{1}{4}}}e^{\omega^2\left(\frac{cbk-dka}{4b^2k^2+a^2}\right)},
\end{aligned}$$

as shown Figures 2(a) and 2(b).

If we take the matrix parameter $B = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, then we easily get

$$|L_B\{f\}(\omega)| = \frac{1}{\sqrt{2k}}e^{-\frac{\omega^2}{4k}},$$

which shows the above equation is consistent with the norm (modulus) of the Gaussian function in the Fourier transform domain. Figures 3(a) and 3(b) illustrate the norm (modulus) of the Gaussian signal in the linear canonical Fourier domain.

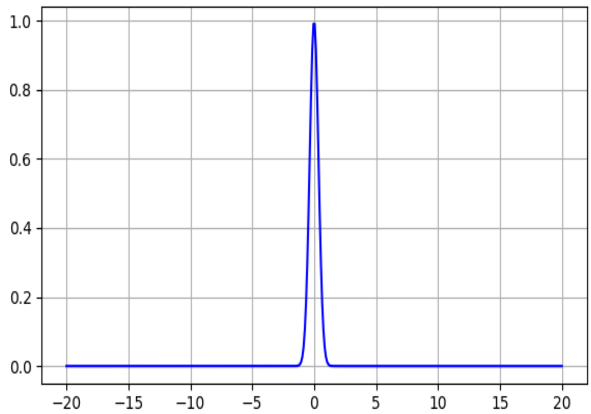


FIGURE 1. Gaussian function for $k = 4$.

Finally, we apply the above relation to evaluate the following function in the linear canonical Fourier domain. Given the function in the following form:

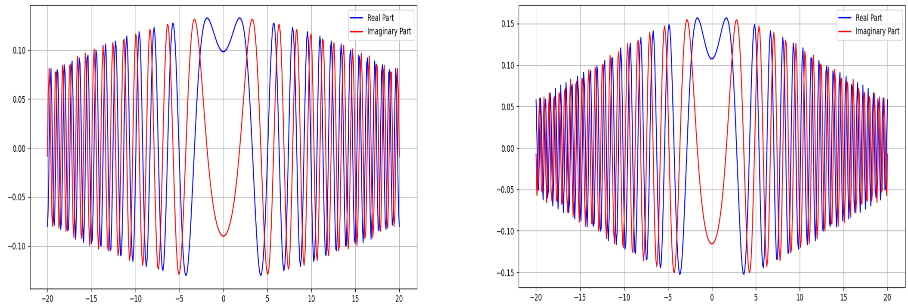
$$f(t) = \begin{cases} 1, & -1 \leq t \leq 1 \\ 0, & t \text{ elsewhere.} \end{cases}$$

In fact, we have

$$\begin{aligned} \mathcal{F}\{h\}(\omega) &= \int_{-\infty}^{\infty} h(t) e^{-i\omega t} dt \\ &= \int_{-\infty}^{\infty} f(t) e^{\frac{ia}{2b} t^2} e^{-i\omega t} dt \\ &= \int_{-1}^{1} e^{\frac{ia}{2b}\left(\left(t - \frac{b}{a}\omega\right)^2 - \frac{b^2\omega^2}{a^2}\right)} dt \\ &= \int_{-1}^{1} e^{\frac{ia}{2b}\left(\left(t - \frac{b}{a}\omega\right)^2\right)} e^{-\frac{ib\omega^2}{2a}} dt \\ &= e^{-\frac{ib\omega^2}{2a}} \int_{-1}^{1} e^{\frac{ia}{2b}\left(\left(t - \frac{b}{a}\omega\right)^2\right)} dt. \end{aligned}$$

Letting $u = t - \frac{b}{a}\omega$, we further obtain

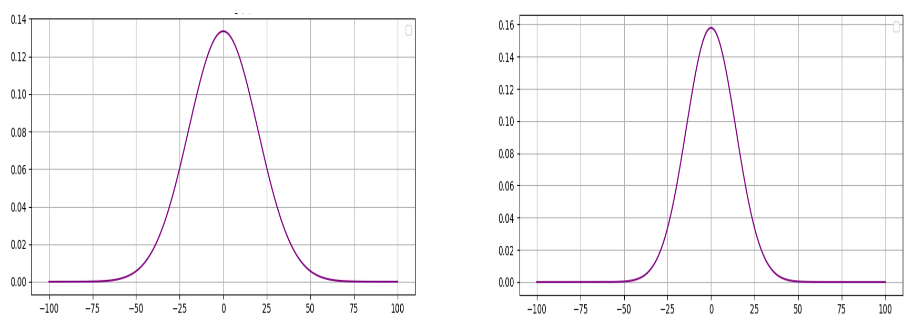
$$\begin{aligned} &\mathcal{F}\{h\}(\omega) \\ &= e^{-\frac{ib\omega^2}{2a}} \int_{-1-\frac{b}{a}\omega}^{1-\frac{b}{a}\omega} e^{\frac{ia}{2b} u^2} du \\ &= e^{-\frac{ib\omega^2}{2a}} \int_{-1-\frac{b}{a}\omega}^{1-\frac{b}{a}\omega} e^{\left(\sqrt{\frac{ia}{2b}} u\right)^2} du \\ &= e^{-i\left(\frac{b\omega^2}{2a} + \frac{\pi}{4}\right)} \sqrt{\frac{2\pi b}{4a}} \left(\operatorname{erf}\left(\sqrt{\frac{ia}{2b}} \left(1 - \frac{b}{a}\omega\right)\right) - \operatorname{erf}\left(\sqrt{\frac{ia}{2b}} \left(-1 - \frac{b}{a}\omega\right)\right)\right). \end{aligned}$$



(a) Gaussian function in LCT domain for $k = 4$ and $B = \begin{bmatrix} 5 & 2 \\ 7 & 3 \end{bmatrix}$.

(b) Gaussian function in LCT domain for $k = 4$ and $B = \begin{bmatrix} -3 & -2 \\ 5 & 3 \end{bmatrix}$.

FIGURE 2. Gaussian functions in the LCT domain for two different matrices B and fixed $k = 4$.



(a)

(b)

FIGURE 3. The norms of the Gaussian functions in the LCT domain for two different matrices for 3(a) $B = \begin{bmatrix} 5 & 2 \\ 7 & 3 \end{bmatrix}$ and 3(b) $B = \begin{bmatrix} -3 & -2 \\ 5 & 3 \end{bmatrix}$ fixed $k = 4$.

Therefore,

$$\begin{aligned}
&\mathcal{F}\{h\}\left(\frac{\omega}{b}\right) \\
&= e^{-i\left(\frac{b\left(\frac{\omega}{b}\right)^2}{2a}+\frac{\pi}{4}\right)} \sqrt{\frac{2\pi b}{4a}} \left(\operatorname{erf}\left(\sqrt{\frac{ia}{2b}} \left(1 - \frac{b}{a}\left(\frac{\omega}{b}\right)\right)\right) - \right. \\
&\qquad \left. \operatorname{erf}\left(\sqrt{\frac{ia}{2b}} \left(-1 - \frac{b}{a}\left(\frac{\omega}{b}\right)\right)\right)\right) \\
&= e^{-i\left(\frac{\omega^2}{2ba}+\frac{\pi}{4}\right)} \sqrt{\frac{2\pi b}{4a}} \left(\operatorname{erf}\left(\sqrt{\frac{ia}{2b}} \left(1 - \frac{\omega}{a}\right)\right) - \operatorname{erf}\left(\sqrt{\frac{ia}{2b}} \left(-1 - \frac{\omega}{a}\right)\right)\right).
\end{aligned}$$

With the help of equation (3.2), we arrive at

$$\begin{aligned} &L_B\{f\}(\omega) \\ &= \frac{1}{\sqrt{2\pi b}} e^{\frac{i}{2}\left(\frac{d}{b}\omega^2 - \frac{\pi}{2}\right)} e^{-i\left(\frac{\omega^2}{2ba} + \frac{\pi}{4}\right)} \sqrt{\frac{2\pi b}{4a}} \left(\operatorname{erf}\left(\sqrt{\frac{ia}{2b}} \left(1 - \frac{\omega}{a}\right)\right) - \right. \\ &\qquad \left. \operatorname{erf}\left(\sqrt{\frac{ia}{2b}} \left(-1 - \frac{\omega}{a}\right)\right)\right) \\ &= \sqrt{\frac{2}{\pi}} e^{\frac{i}{2}\left(\frac{d}{b}\omega^2 - \frac{2\omega^2}{2ba}\right)} \left(\operatorname{erf}\left(\sqrt{\frac{ia}{2b}} \left(1 - \frac{\omega}{a}\right)\right) - \operatorname{erf}\left(\sqrt{\frac{ia}{2b}} \left(-1 - \frac{\omega}{a}\right)\right)\right), \end{aligned}$$

where

$$\operatorname{erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-t^2} dt.$$

4. Conclusion

In this paper, we introduced the linear canonical transform and its basic relationship to the traditional Fourier transform. We utilized the relations to compute the Gaussian function in the linear canonical transform. In future work, we shall apply the relation to compute other functions in the linear canonical transform domains.

References

[1] M. Bahri, R. Ashino, An Inequality for Linear Canonical Transform, In: S. L. Peng, M. Favorskaya, H. C. Chao (Eds.), *Sensor Networks and Signal Processing*, Smart Innovation, Systems and Technologies, vol 176. Springer, Singapore, 2021.

[2] D. Bing, T. Ran, W. Yue, Convolution theorems for the linear canonical transform and their application, *Sci. China Inf. Sci.* **49(5)** (2006) 592–603.

[3] P. Dang, G. T. Deng, T. Qian, A tighter uncertainty principle for linear canonical transform in terms of phase derivative, *IEEE Trans. Signal Process.* **61(21)** (2013) 5153–5164.

[4] L. Debnath, F. A. Shah, *Wavelet transforms and their applications*, Springer Nature, Switzerland, 2020.

[5] Q. Feng, B. Z. Li, Convolution and correlation theorems for the two-dimensional linear canonical transform and its applications, *IET Sig. Process.* **10(2)** (2016) 125–132.

[6] N. Goel, K. Singh, R. Saxena, Multiplicative filtering in the linear canonical transform domain, *IET Sig. Process.* **10(2)** (2016) 173–181.

[7] Y. Guo, B. Z. Li, Blind image watermarking method based on linear canonical transform and QR decomposition, *IET Sig. Process.* **10(10)** (2016) 773–786.

[8] J. J. Healy, P. O'Grady, J. T. Sheridan, *Simulating paraxial optical systems using the linear canonical transform: properties, issues and applications*, Proc. SPIE 7072, Optics and Photonics for Information Processing II, 70720E, 2008.

[9] Z. J. Huang, S. Cheng, L. H. Gong, N. R. Zhou, Nonlinear optical multi-image encryption scheme with two-dimensional linear canonical transform, *Opt. Lasers Engg.* **124** (2020) 105821.

[10] D. Wei, Filter bank reconstruction of band-limited signals from multichannel samples associated with the LCT, *IET Sig. Process* **11(3)** (2016) 320–331.
[11] D. Wei, H. Hu, Sparse discrete linear canonical transform and its application, *Signal Process.* **183** (2021) 108046.
[12] D. Wei, Q. Ran, Y. Li, A convolution and convolution theorem for the linear canonical transform and its application, *Circuits Syst. Sig. Process.* **31(1)** (2012) 301–312.
[13] D. Wei, Q. Ran, Y. Li, New convolution theorem for the linear canonical transform and its translation invariance property, *Optik-Int. J. Light Electron. Opt.* **123(16)** (2012) 1478–1481.
[14] Z. C. Zhang, Jittered sampling in linear canonical domain, *IEEE Commun. Lett.* **24(7)** (2020) 1529–1523.