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# On the angle between two subspaces of an inner product space

**Hendra Gunawan**[1,*]

[1] *Analysis and Geometry Group, Faculty of Mathematics and Natural Sciences*
*Bandung Institute of Technology, Jl. Ganesha No.10, Bandung 40132, Indonesia.*
*e-mail: hgunawan@itb.ac.id.*

**Abstract** In this paper, we shall discuss two formulas for the angle between a subspace of dimension $p$ and another subspace of dimension $q$ with $1 \leq p \leq q \leq \dim(X)$, in an inner product space $X$. In particular, we shall see that the two seemingly different formulas, one is defined by H. Gunawan, O. Neswan, and W. Setya-Budhi in 2005 and the other by I. B. Risteski and K. G. Trenčevski in 2001 and N. Wildberger in 2017, are actually identical.

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## 1. Introduction

Let $(X, \langle\cdot,\cdot\rangle)$ be a real inner product space, $U := \text{span}\{u_1, \ldots, u_p\}$ be a $p$-dimensional subspace, and $V := \text{span}\{v_1, \ldots, v_q\}$ be a $q$-dimensional subspace of $X$, where $1 \leq p \leq q \leq \dim(X)$. As in [3], the angle $\theta := \theta(U, V)$ between the subspace $U$ and the subspace $V$ may be defined by

$$\cos\theta := \frac{\|\text{proj}_V u_1, \ldots, \text{proj}_V u_p\|}{\|u_1, \ldots, u_p\|}, \tag{1.1}$$

where $\|u_1, \ldots, u_p\|$ denotes the volume of the $p$-dimensional parallelepiped spanned by $u_1, \ldots, u_p$, given by

$$\|u_1, \ldots, u_p\| := \sqrt{\det[\langle u_i, u_j\rangle]_{p\times p}}.$$

Note that the value of $\cos\theta$ here is non-negative. This means that we always choose the acute angle between the two subspaces.

For $n \in \mathbb{N}$ in general, the expression $\|x_1, \ldots, x_n\|$ is known as the 'standard' $n$-norm on $X$, which was studied in [2]. It is clear that $\|x_1, \ldots, x_n\| \geq 0$ and $\|x_1, \ldots, x_n\| = 0$ if

*Corresponding author.



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and only if $x_1, \dots, x_n$ are linearly dependent. Moreover, $\|x_1, \dots, x_n\|$ is invariant under permutation, and

$$\|\alpha x_1, x_2, \dots, x_n\| = |\alpha| \, \|x_1, \dots, x_n\|$$

for every $\alpha \in \mathbb{R}$ and $x_1, \dots, x_n \in X$. Also,

$$\|x_0 + x_1, x_2, \dots, x_n\| \le \|x_0, x_2, \dots, x_n\| + \|x_1, x_2, \dots, x_n\|$$

for every $x_0, x_1, x_2, \dots, x_n \in X$. See [2] for more properties of the standard $n$-norm.

Along with the the standard $n$-norm, we have the standard $n$-inner product on $X$ [1]:

$$\langle x_0, x_1 | x_2, \dots, x_n \rangle := \begin{vmatrix} \langle x_0, x_1 \rangle & \langle x_0, x_2 \rangle & \cdots & \langle x_0, x_n \rangle \\ \langle x_2, x_1 \rangle & \langle x_2, x_2 \rangle & \cdots & \langle x_2, x_n \rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle x_n, x_1 \rangle & \langle x_n, x_2 \rangle & \cdots & \langle x_n, x_n \rangle \end{vmatrix}.$$

Note particularly that $\langle x_1, x_1 | x_2, \dots, x_n \rangle = \det[\langle x_i, x_j \rangle] = \|x_1, x_2, \dots, x_n\|^2$.

Now, with the properties of the $p$-norm on $X$, we can observe some basic properties of the angle $\theta$ given via the formula for $\cos\theta$ in (1.1):

- The ratio that defines $\cos\theta$ is indeed a number in $[0, 1]$. In particular, if the projection of one of the vectors spanning $U$ on $V$ is a zero vector, then $\cos\theta = 0$ or $U$ is orthogonal to $V$;
- The projection of $u_i$'s on $V$ is independent of the choice of basis for $V$ (thus, for example, we may replace $v_1$ and $v_2$ by $v_1 + v_2$ and $v_1 - v_2$);
- The projections are linear transformations, and so the ratio that defines $\cos\theta$ is also invariant under any change of basis for $U$.

For a special case when $p = q$, it can be shown that

$$\cos^2\theta = \frac{\det(MM^{\mathrm{T}})}{\det[\langle u_i, u_j \rangle] \det[\langle v_k, v_l \rangle]} = \frac{(\det M)^2}{\det[\langle u_i, u_j \rangle] \det[\langle v_k, v_l \rangle]}, \tag{1.2}$$

where $M := [\langle u_i, v_k \rangle]$ [3, 4].

## 2. Main Results

### 2.1. The first formula

We can derive a general formula for the angle $\theta$ between the subspace spanned by $U$ (of dimension $p$) and that by $V$ (of dimension $q$) as in [3]. By writing

$$\mathrm{proj}_V u_j := \sum_{k=1}^{q} \alpha_{jk} v_k,$$

where (by the Cramer's rule)

$$\alpha_{jk} = \frac{\langle u_j, v_k | v_{i_2(k)}, \dots, v_{i_q(k)} \rangle}{\|v_1, \dots, v_q\|^2} = \frac{\langle u_j, v_k | v_{i_2(k)}, \dots, v_{i_q(k)} \rangle}{\det[\langle v_k, v_l \rangle]},$$

with $\{i_2(k), \dots, i_q(k)\} = \{1, 2, \dots, q\} \setminus \{k\},\ k = 1, 2, \dots, q$, we have

$$\langle \mathrm{proj}_V u_i, \mathrm{proj}_V u_j \rangle = \langle u_i, \mathrm{proj}_V u_j \rangle = \sum_{k=1}^{q} \alpha_{jk} \langle u_i, v_k \rangle,$$

for $i, j = 1, \ldots, p$. Next, computing

$$\|\mathrm{proj}_V u_1, \ldots, \mathrm{proj}_V u_p\|^2 = \begin{vmatrix} \sum_{k=1}^{q} \alpha_{1k}\langle u_1, v_k\rangle & \cdots & \sum_{k=1}^{q} \alpha_{pk}\langle u_1, v_k\rangle \\ \vdots & \ddots & \vdots \\ \sum_{k=1}^{q} \alpha_{1k}\langle u_p, v_k\rangle & \cdots & \sum_{k=1}^{q} \alpha_{pk}\langle u_p, v_k\rangle \end{vmatrix}$$
$$= \frac{\det(M\widetilde{M}^{\mathrm{T}})}{\det^p[\langle v_k, v_l\rangle]},$$

where $M; = [\langle u_i, v_k\rangle]$ and $\widetilde{M} := [\langle u_i, v_k | v_{i_2(k)}, \ldots, v_{i_q(k)}\rangle]$, with $\{i_2(k), \ldots, i_q(k)\}$ as before, the formula (1.1) becomes

$$\cos^2\theta = \frac{\det(M\widetilde{M}^{\mathrm{T}})}{\det[\langle u_i, u_j\rangle]\det^p[\langle v_k, v_l\rangle]}, \tag{2.1}$$

as obtained in [3]. This is the first formula that we derive from (1.1).

## 2.2. The second formula

Let $X = \mathbb{R}^d$, equipped with the usual inner product, $U = \{u_1, \ldots, u_p\}$ be a $p$-dimensional subspace of $X$, and $V = \{v_1, \ldots, v_q\}$ be a $q$-dimensional subspace of $X$ with $1 \le p \le q \le d$. Next, let $P$ be the $p \times d$ matrix whose $i$-th row is $u_i$ $(i = 1, \ldots, p)$ and $Q$ be the $q \times d$ matrix whose $k$-th row is $v_k$ $(k = 1, \ldots, q)$. Then, for $p = q$, the angle $\theta$ between $U$ and $V$ is given by

$$\cos^2\theta := \det\big(PQ^{\mathrm{T}}(QQ^{\mathrm{T}})^{-1}QP^{\mathrm{T}}(PP^{\mathrm{T}})^{-1}\big). \tag{2.2}$$

Note that, in general, $PQ^{\mathrm{T}}$ is of size $p \times q$ (and so $QP^{\mathrm{T}}$ is of size $q \times p$), $PP^{\mathrm{T}}$ is of size $p \times p$, and $QQ^{\mathrm{T}}$ is of size $q \times q$. The above formula originated in [4], which was later explored in [5].

As we shall see below, the formula (2.2) is identical to the formula (2.1), by showing that it can also be derived from (1.1). In addition, we can also generalize the formula (2.2) in an infinite-dimensional inner product space, just like the formula (2.1).

## 2.3. The two formulas are the same

Assume here that $X = \mathbb{R}^d$, equipped with the usual inner product denoted by $\langle\cdot,\cdot\rangle$, and $1 \le p \le q \le d$. Let us first consider the case where $p = q$, where $PP^{\mathrm{T}}$, $PQ^{\mathrm{T}}$, $QP^{\mathrm{T}}$, and $QQ^{\mathrm{T}}$ are square matrices. In this case, the formula (2.2) reduces to

$$\cos^2\theta = \frac{\det(PQ^{\mathrm{T}})\det(QP^{\mathrm{T}})}{\det(QQ^{\mathrm{T}})\det(PP^{\mathrm{T}})} = \frac{(\det M)^2}{\det[\langle u_i, u_j\rangle]\det[\langle v_k, v_l\rangle]},$$

where $M := [\langle u_i, v_k\rangle]$, the same as (1.2).

For $1 \le p \le q \le d$ in general, we have the following theorem, which tells us that the two formulas (2.1) and (2.2) are identical.

**Theorem 2.1.** *The formula (2.2) is the same as the formula (2.1).*

*Proof.* For each $j \in \{1, \ldots, p\}$, let $u_{jV} = \sum_{k=1}^{q} \alpha_{jk} v_k = \mathrm{proj}_V u_j$, namely the projection of $u_j$ on $V$. Then, we have

$$u_j - u_{jV} \perp V,$$

which implies that

$$\langle v_l, u_{jV} \rangle = \langle v_l, u_j \rangle, \quad l = 1, \ldots, q.$$

Hence

$$\sum_{k=1}^{q} \alpha_{jk} \langle v_l, v_k \rangle = \langle v_l, u_j \rangle, \quad l = 1, \ldots, q.$$

From the last equations, we get

$$\begin{bmatrix} \langle v_1, v_1 \rangle & \cdots & \langle v_1, v_q \rangle \\ \vdots & \ddots & \vdots \\ \langle v_q, v_1 \rangle & \cdots & \langle v_q, v_q \rangle \end{bmatrix} \begin{bmatrix} \alpha_{j1} \\ \vdots \\ \alpha_{jq} \end{bmatrix} = \begin{bmatrix} \langle v_1, u_j \rangle \\ \vdots \\ \langle v_q, u_j \rangle \end{bmatrix},$$

whence

$$\begin{bmatrix} \alpha_{j1} \\ \vdots \\ \alpha_{jq} \end{bmatrix} = \begin{bmatrix} \langle v_1, v_1 \rangle & \cdots & \langle v_1, v_q \rangle \\ \vdots & \ddots & \vdots \\ \langle v_q, v_1 \rangle & \cdots & \langle v_q, v_q \rangle \end{bmatrix}^{-1} \begin{bmatrix} \langle v_1, u_j \rangle \\ \vdots \\ \langle v_q, u_j \rangle \end{bmatrix} = (QQ^{\mathrm{T}})^{-1} (QP^{\mathrm{T}})_j,$$

where $(QP^{\mathrm{T}})_j$ denotes the $j$-th column of $QP^{\mathrm{T}}$. Now $\|u_{1V}, \ldots, u_{pV}\|^2 = \det[\langle u_{iV}, u_{jV} \rangle]$. But

$$\langle u_{iV}, u_{jV} \rangle = \langle u_i, u_{jV} \rangle = \sum_{k=1}^{q} \alpha_{jk} \langle u_i, v_k \rangle,$$

for $i = 1, \ldots, p$. Hence

$$\begin{aligned} \|u_{1V}, \ldots, u_{pV}\|^2 &= \begin{vmatrix} \sum_{k=1}^{q} \alpha_{1k} \langle u_1, v_k \rangle & \cdots & \sum_{k=1}^{q} \alpha_{pk} \langle u_1, v_k \rangle \\ \vdots & \ddots & \vdots \\ \sum_{k=1}^{q} \alpha_{1k} \langle u_p, v_k \rangle & \cdots & \sum_{k=1}^{q} \alpha_{pk} \langle u_p, v_k \rangle \end{vmatrix} \\ &= \left| [\langle u_i, v_k \rangle]_{p \times q} \cdot A_{q \times p} \right|, \end{aligned}$$

where $A_{q \times p} = [\alpha_{kj}] = (QQ^{\mathrm{T}})^{-1} QP^{\mathrm{T}}$.

We therefore arrive at the conclusion that

$$\frac{\|u_{1V}, \ldots, u_{pV}\|^2}{\|u_1, \ldots, u_p\|^2} = \frac{\det(PQ^{\mathrm{T}}(QQ^{\mathrm{T}})^{-1} QP^{\mathrm{T}})}{\det(PP^{\mathrm{T}})},$$

which implies that the formula (2.2) is identical with the formula (2.1). ∎

### 2.4. The infinite-dimensional case

We notice that in deriving the formula (2.2) from (1.1), the crucial step is finding the coefficients of the projection vectors $u_{jV}$'s. The dimension of the space is absorbed in the inner products $\langle v_k, v_l\rangle$'s and $\langle u_i, v_k\rangle$'s. With these remarks, we may replace the matrices $PQ^{\mathrm{T}}$, $QQ^{\mathrm{T}}$, and $PP^{\mathrm{T}}$ by $[\langle u_i, v_k\rangle]$, $[\langle v_k, v_l\rangle]$, and $[\langle u_i, u_j\rangle]$ respectively, and obtain the formula

$$\cos^2\theta = \frac{\det\big(M[\langle v_k, v_l\rangle]^{-1}M^{\mathrm{T}}\big)}{\det[\langle u_i, u_j\rangle]}, \tag{2.3}$$

where $M = [\langle u_i, v_k\rangle]$. Note that if $p = q$, then (2.3) reduces back to

$$\cos^2\theta = \frac{(\det M)^2}{\det[\langle u_i, u_j\rangle]\det[\langle v_k, v_l\rangle]},$$

as before.

Thus, to summarize, by replacing $PQ^{\mathrm{T}}$, $QQ^{\mathrm{T}}$, and $PP^{\mathrm{T}}$ by $[\langle u_i, v_k\rangle]$, $[\langle v_k, v_l\rangle]$, and $[\langle u_i, u_j\rangle]$ respectively, we have seen that the formula (2.2) extends to any inner product space $(X, \langle\cdot,\cdot\rangle)$, including the infinite-dimensional ones, as does the formula (2.1).

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