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Some Convergence Results for Split Common and Fixed Point Problems in Hilbert Spaces

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Abstract In this survey article, we present an introduction to split feasibility problems, multiple-set split feasibility problems, and split common fixed point problems. Parallel and cyclic algorithms for solving the split common fixed point problems for a finite family of strictly pseudo contractive mappings in Hilbert spaces are presented, and also weak and strong convergence theorems are proved. In the end, applications of the split common fixed point problem and some numerical examples are also presented.

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1. Introduction

Throughout this paper, we consider $\langle .,. \rangle$ to be an inner product and $\|.\|$ as its corresponding norm. Let $H_i, i = 1, 2,$ be Hilbert spaces, $C_i, i = 1, 2, 3, ..., p$ and $Q_j, j = 1, 2, 3, ..., r$ be nonempty, closed, and convex subsets of H_1 and H_2, respectively.

The convex feasibility problem (CFP) is obtained as finding a vector $x^* \in H_1$ satisfying:

$$x^* \in \bigcap_{i=1}^{p} C_i. \tag{1.1}$$

This problem had intensively being studied by a number of authors due to its various applications in real life problems, such as in approximation theorem, image recovery,

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signal processing, control theory, biomedical engineering, communication, and geophysics, see[4, 13, 24]. As a generalization of CFP, we have multiple-set split feasibility problem (MSSFP). This is formulated as finding a vector $x^* \in H_1$ with the property:

$$x^* \in \bigcap_{i=1}^{p} C_i \quad \text{such that} \quad Ax^* \in \bigcap_{j=1}^{r} Q_j, \tag{1.2}$$

where A is a bounded and linear mapping. Set $p = r = 1$, we have

$$x^* \in C \quad \text{such that} \quad Ax^* \in Q. \tag{1.3}$$

Equation (1.3) is known as the split feasibility problem (SFP), see Censor and Segal [12].

Since every nonempty, closed, and convex subset of a Hilbert space is a fixed point of its associating projection, then Equations (1.1) and (1.2) give

$$x^* \in \bigcap_{i=1}^{p} Fix(U_i), \tag{1.4}$$

and

$$x^* \in \bigcap_{i=1}^{p} Fix(U_i) \quad \text{such that} \quad Ax^* \in \bigcap_{j=1}^{r} Fix(T_j). \tag{1.5}$$

where $U_i : H_1 \to H_1$ $(i = 1, 2, 3, \ldots, p)$ and $T_j : H_2 \to H_2$ $(j = 1, 2, 3, \ldots, r)$ are some nonlinear operators.

Equations (1.4) and (1.5) are called "common fixed point problem (CFPP) and split common fixed point problem (SCFPP)", respectively,

Set $p = r = 1$, Equation (1.5) reduces to

$$x^* \in Fix(U) \quad \text{such that} \quad Ax^* \in Fix(T). \tag{1.6}$$

This is called a two-set SCFPP.

1.1. Split Feasibility Problem (SFP)

The idea of SFP was established in 1994 by Censor and Elfving [9]. The SFP attracts the attention of many researchers due to its various applications in many physical problems, such as in intensity modulation radiation therapy, signal processing, and image reconstruction, see [4, 13, 24]. Subsequently, a number of iterative methods for solving SFP for many nonlinear mappings in Hilbert spaces were established, for more details, see [7, 8, 10, 19, 25, 26]. It is well-known that to solve SFP, the computation of the inverse of a bounded linear operator is necessary, this was why Byne [7], considered an algorithm for solving such a problem that does not include the inverse of a bounded linear operator. Similarly, in 2002, Byne [7], also introduced another algorithm known as CQ-algorithm for solving SFP and obtained a weak convergence result. To solve the CQ-algorithm, the computation of metric projections on C and Q are necessary, and this is not an easy task in practice. More results on CQ-algorithm for solving SFP can be found in [14, 22, 23]. It is well-known that SFP can be reduced to convex feasibility problem (CFP) as well as fixed point problem, see Mohammed and Kılıçman [20].

1.2. Multiple-Sets Split Feasibility Problem

The multiple-sets split feasibility problem (in short, MSSFP) can be used as a model for many inverse problems where constraints are imposed on the solutions in the domain of a linear operator as well as in the operator's range. The MSSFP generalizes the convex feasibility problem and split feasibility problem as can be found in [20]. The MSSFP arises in the field of intensity-modulated radiation therapy (in short, IMRT) when one attempts to describe physical dose constraints and equivalent uniform dose (EUD) within a single model, see Censor et al [11]. The IMRT is an advanced mode of high-precision radiotherapy that uses computer-controlled linear accelerators to deliver precise radiation doses to specific areas within a tumor.

The IMRT allows for the radiation doses to confirm the three-dimensional (3D) shape of the tumor more precisely by modulating or controlling the intensity of the radiation beam in multiple small volumes. It also allows higher radiation doses to be focused on regions within the tumor while minimizing the dose to surrounding normal critical on structures. Treatment is carefully planned by using 3D computed tomograpy (CT) or magnetic resonance images of the patient in conjunction with computerized dose calculations to determine the dose intensity pattern that will confirm the tumor shape best. Typically, combinations of multiple intensity-modulated field coming from different beam directions produce a customized radiation dose that maximizes the dose to the targeted tumor while minimizing the dose to adjacent normal tissues. The ratio of normal tissue dose to tumor dose is reduced to a minimum with the IMRT approach. Higher and more effective radiation doses can safely be delivered to tumor with fewer side effects compared with conventional radiotherapy techniques. The IMRT also has the potential to reduce treatment toxicity, even when doses are not increased. Radiation therapy, including IMRT stops cancer cells from dividing and growing, thus, it is slowing or stopping tumor growth. In many cases, radiation therapy is capable of killing all of the cancer cells, thus shrinks or eliminates tumors, for more details, see Ansari and Rehann [2].

1.3. Split Common Fixed Point Problem (SCFPP)

The idea of the SCFPP was developed in 2009 by Censor and Segal (see [12]) and it entails finding a vector of the family of an operator in one space such that its image under a bounded linear operator is a common fixed point of another family of an operator in the image space. The SCFPP generalizes the convex feasibility problem (CFP), the multiple-set split feasibility problem (MSSFP), and the split feasibility problem (SFP), for more details, see [20] and the reference therein. Censo and Segal (see [12]), had considered the following algorithm:

$$x_{n+1} := U\Big(x_n + \gamma A^*(T - I)Ax_n\Big), \forall n \geq 0, \tag{1.7}$$

where the initial guess $x_0 \in H$ is chosen arbitrarily and $0 < \gamma < \frac{2}{\|A\|^2}$, T and S are demicontractive mappings. Based on the work of Censor and Segal (see [12]), Moudafi [21] had studied the convergence properties of relaxed algorithm for solving the SCFPP for a class of quasi-nonexpansive operator T such that $(I - T)$ is demiclosed at zero and obtained a weak convergence result. Note that, in finite-dimensional Hilbert space, weak and strong convergence are equivalent. It is different in an infinite dimensional space, that is, they are not the same. Moudafi's results guarantee only a weak convergence result. Based on this, Mohammed [17, 18] utilized the strongly quasi nonexpansive operators and quasi nonexpansive operators to solve Moudafi's algorithm and obtained strong convergence results.

Kraikaew and Saejung [15] also modified the Moudafi's algorithm [21] and obtained a strong convergence result as stated below.

Theorem 1.1. *(Kraikaew and Saejung [15]) let $U : H_1 \to H_1$ be a strongly quasi-nonexpansive operator and $T : H_2 \to H_2$ be a quasi-nonexpansive operator such that both $(I - U)$ and $(I - T)$ are demiclosedness at zero. Let $A : H_1 \to H_2$ be a bounded linear operator with $L = \|AA^*\|$. Suppose that $\Gamma \neq \emptyset$. Let $\{x_n\} \subset H_1$ be a sequence generated by*

$$\begin{cases} x_0 \in H, \\ x_{n+1} = \alpha_n x_0 + (1 - \alpha_n)U(x_n + \gamma A^*(T - I)Ax_n), \end{cases} \tag{1.8}$$

where the parameter γ and $\{\alpha_n\}$ satisfy the following conditions:

(a) $\gamma \in (0, \frac{1}{L})$;
(b) $\{\alpha_n\} \subset (0, 1), \lim_{n\to\infty} \alpha_n = 0$ and $\sum \alpha_n = \infty$. Then $x_n \to P_\Gamma x_0$.

One of important classes of nonlinear mappings is the class of strictly pseudocontractive mapping. This mapping is defined as

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + k \|(I - T)x - (I - T)y\|^2, \forall x, y \in H_1, \tag{1.9}$$

where $T : H_1 \to H_1$, and $k \in [0, 1)$. If $k = 0$, Equation (1.9) reduces to

$$\|Tx - Ty\| \leq \|x - y\|. \tag{1.10}$$

This is known as nonexpansive mapping.

Numerous authors have extensively studied iterative algorithms for strictly pseudo-contractive mapping; for example, see [1, 3, 6, 16, 27–29] and the references therein. For instance, in 1967, Browder and Petryshyn [5] proved that if U is k-strictly pseudocontractive which has a fixed point in C, then the sequence $\{x_n\}$ generated by

$$\begin{cases} x_0 \text{ is chosen arbitrarily}, \\ x_{n+1} = \alpha x_n + (1 - \alpha)Ux_n, \end{cases} \tag{1.11}$$

converges weakly to the fixed point of U provided that $\alpha \in (k, 1)$. Thereafter, Marino and Xu [16] modified Algorithm (1.11) and considered the following algorithm:

$$\begin{cases} x_0 \text{ is chosen arbitrarily}, \\ x_{n+1} = \alpha_n x_n + (1 - \alpha_n)Ux_n. \end{cases} \tag{1.12}$$

It was proved by Marino and Xu[16] that the sequence $\{x_n\}$ generated by Algorithm (1.12) converges weakly to the common fixed point of U provided that $\{\alpha_n\}$ satisfies the following conditions:

$$k < \alpha_n < 1 \text{ for all } n \text{ and } \sum_{n=0}^{\infty} (\alpha_n - k)(1 - \alpha_n) = \infty. \tag{1.13}$$

In 2007, Acedo and Xu [1] considered the following algorithm:

$$\begin{cases} x_0 \text{ is chosen arbitrarily}, \\ x_{n+1} = \alpha_n x_n + (1 - \alpha_n)\sum_{s=1}^{N} \lambda_s U_s x_n, \forall n \geq 0, \end{cases} \tag{1.14}$$

where $\lambda_s > 0$, $\forall s$ positive integer, $\sum_{s=1}^{N} \lambda_s = 1$, $\{U_s\}_{s=1}^{N}$ is k-strictly pseudocontractive mappings and $\{\alpha_n\}$ is a sequence in (0,1) satisfied some certain conditions, see [1]. It was proved that the sequence $\{x_n\}$ generated by Algorithm 1.14 converges weakly to a solution of Problem (1.4).

Motivated by these results, in this paper, we construct parallel and cyclic algorithms of a finite family of strictly pseudocontractive mappings to solve the following split common fixed point problems:

$$\text{Find } x^* \in \bigcap_{s=1}^{N} Fix(U_s) \text{ such that } Ax^* \in \bigcap_{r=1}^{M} Fix(T_r), \tag{1.15}$$

where $N \geq 1$ and $M \geq 1$ are positive integers, $\{U_s\}_{s=1}^N$ and $\{T_r\}_{r=1}^M$ are N and M strictly pseudocontractive mappings.

Define U and T by

$$U = \sum_{s=1}^{N} \lambda_s U_s \quad \text{and } T = \sum_{r=1}^{M} \beta_r T_r, \tag{1.16}$$

where s and r are positive integers, $\lambda_s > 0$, $\beta_r > 0$, $\sum_{s=1}^N \lambda_s = 1$, and $\sum_{r=1}^M \beta_r = 1$. We will show that U and T are strictly pseudocontractive mappings, $Fix(U) = \bigcap_{s=1}^N Fix(U_s)$ and $Fix(T) = \bigcap_{r=1}^M Fix(T_r)$. We will also show that the sequence $\{x_n\}$ defined by

$$\begin{cases} x_0 \in H_1 \text{ is chosen arbitrarily}, \\ u_n = x_n + \gamma A^*\big(\sum_{r=1}^M \beta_r T_r - I\big)Ax_n, \\ x_{n+1} = \alpha_n u_n + (1-\alpha_n)\sum_{s=1}^N \lambda_s U_s u_n, \forall n \geq 0, \end{cases} \tag{1.17}$$

converges weakly to the solution of Problem (1.15).

In Algorithm (1.17), the sequences $\{\beta_r\}_{r=1}^M$ and $\{\lambda_s\}_{s=1}^N$ are constants in the sense that both are independent of n; thus, we will consider a more general case in which $\{\beta_r\}_{r=1}^M$ and $\{\lambda_s\}_{s=1}^N$ depend on n. Hence, we consider the following iterative algorithm which generates a sequence $\{x_n\}$ by

$$\begin{cases} x_0 \in H_1 \text{ is chosen arbitrarily}, \\ u_n = x_n + \gamma A^*\big(\sum_{r=1}^M \beta_r^n T_r - I\big)Ax_n, \\ x_{n+1} = \alpha_n u_n + (1-\alpha_n)\sum_{s=1}^N \lambda_s^n U_s u_n, \forall n \geq 0. \end{cases} \tag{1.18}$$

By imposing some appropriate conditions on $\{\beta_r\}_{r=1}^M$ and $\{\lambda_s\}_{s=1}^N$, we will also show that the sequence $\{x_n\}$ generated by Algorithm (1.18) will converge weakly to the solution of Problem (1.15).

Another approach to the Problem (1.15) is the cyclic algorithm:

Let x_0 be arbitrarily chosen, define a sequence $\{x_n\}$ cyclically by;

$$\begin{cases} u_0 = x_0 + \gamma A^*\big(T_0 - I\big)Ax_0, \\ x_1 = \alpha_0 u_0 + (1-\alpha_0)U_0 u_0, \end{cases}$$

$$\begin{cases} u_1 = x_1 + \gamma A^*\big(T_1 - I\big)Ax_1, \\ x_2 = \alpha_1 u_1 + (1-\alpha_1)U_1 u_1, \end{cases}$$

$$\vdots$$

$$\begin{cases} u_N = x_N + \gamma A^*\big(T_0 - I\big)Ax_N, \\ x_{N+1} = \alpha_N u_N + (1-\alpha_N)U_0 u_N, \end{cases}$$

$$\vdots$$

In a more compact form, the sequence $\{x_n\}$ can be written as

$$\begin{cases} u_n = x_n + \gamma A^*(T_{[n]} - I)Ax_n, \\ x_{n+1} = \alpha_n u_n + (1-\alpha_n)U_{[n]}u_n, \end{cases} \tag{1.19}$$

where $[n] := n(mod\, N)$ with the mod function taking values in $\{1, 2, 3, ..., N\}$. We will also show that Algorithm (1.19) converges weakly to the solution of Problem (1.15).

In what follows, we adopt the notations; I to be the identity operator on H_1, $Fix(U)$ to be the fixed point set of U i.e., $Fix(U) = \{x \in H_1 : Ux = x\}$, " $\to$ " and " $\rightharpoonup$ " to denote the strong and weak convergence, respectively, and $\omega_\omega(x_n)$ to denote the set of the cluster point of $\{x_n\}$ in the weak topology, i.e., {there exists a subsequence say $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \rightharpoonup x$}, and Γ to denote the solution set of SCFPP (1.15). i.e.,

$$\Gamma = \Big\{x^* \in C := \bigcap_{s=1}^{N} Fix(U_s) \text{ such that } Ax^* \in Q := \bigcap_{r=1}^{M} Fix(T_r)\Big\}. \tag{1.20}$$

2. Preliminaries

The following definitions and lemmas were used in proving our main results.

Definition 2.1. Let C be a nonempty closed convex subset of a Hilbert space (H). A sequence $\{x_n\} \in H$ is called Fejer monotone with respect to C if

$$\|x_{n+1} - x\| \le \|x_n - x\|, \qquad \forall\, x \in C.$$

Definition 2.2. Let H be a Hilbert space. A mapping $T : H \to H$ is said to be demiclosed at zero, if for any sequence $\{x_n\} \in H$, there exists $x \in H$ such that if $x_n \rightharpoonup x$ and $Tx_n \to 0$, then $Tx = 0$.

Lemma 2.3. *(Bauschke and Borwein [4]) Let $\{x_n\}$ be a Fejer monotone with respect to a nonempty closed convex subset C, then $\{P_C x_n\}$ converges strongly. Moreover, $x_n \rightharpoonup x^* \in C$ if and only if $\omega_\omega(x_n) \subset C$.*

Lemma 2.4. *(Acedo and Xu [1]) For each $x, y \in H_1$, the following results hold.*

(i) $\|x+y\|^2 = \|x\|^2 + 2\langle x, y\rangle + \|y\|^2$,
(ii) $\|\alpha x + (1-\alpha)y\|^2 = \alpha\|x\|^2 + (1-\alpha)\|y\|^2 - \alpha(1-\alpha)\|x-y\|^2$, $\forall\, \alpha \in [0,1]$.

Lemma 2.5. *(Marino and Xu [16]) Let C be a nonempty closed convex subset of a Hilbert space H_1 and $T : C \to C$ be a k-strictly pseudocontractive.*
Then the following results hold:

(i) $\|Tx - Ty\| \le \frac{1+k}{1-k}\|x-y\|, \forall x, y \in C$. *i.e. T satisfies the Lipschitz condition.*
(ii) $(T - I)$ *is demiclosed at zero.*
(iii) Given an integer $M \ge 1$, assume for each $1 \le j \le M$, $T_j : C \to C$ is k_j-strictly pseudocontractive for some $k_j \in [0,1)$. Suppose that $\{\lambda_j\}_{j=1}^M$ is a positive sequence such that $\sum_{j=1}^M \lambda_j = 1$, and let $T = \sum_{j=1}^M \lambda_j T_j$.
Then, T is $k-$strictly pseudocontractive with $k = \max\{k_j : 1 \le j \le M\}$.
(iv) Let $\{T_j\}_{j=1}^M$, $\{\lambda_j\}_{j=1}^M$ and T be as in (iii) *above. Assume that $\{T_j\}_{j=1}^M$ has a common fixed point. Then*

$$Fix(T) = \bigcap_{j=1}^{M} Fix(T_j).$$

Lemma 2.6. *Let $T : H_1 \to H_1$ be k-strictly pseudocontractive mapping. Assume that the $Fix(T) \neq \emptyset$. Then, for each $x \in H_1$ and $p \in Fix(T)$, the following inequalities are equivalent.*

(i) $\|Tx - p\|^2 \leq \|x - p\|^2 + k \|Tx - x\|^2$,

(ii) $2 \langle Tx - x, x - p \rangle \leq -(1-k) \|Tx - x\|^2$,

(iii) $2 \langle Tx - x, Tx - p \rangle \leq (1+k) \|Tx - x\|^2$.

Lemma 2.7. *(Acedo and Xu [1]) Let C be a nonempty closed convex subset of a Hilbert space H_1 and P_C be a metric projection from H_1 onto C. Then, $\forall y \in C$* and $x \in H_1$,

$$\|x - P_C(x)\|^2 \leq \|y - x\|^2 - \|y - P_C(x)\|^2.$$

Lemma 2.8. *(Xu [27]) Let $\{a_n\}$ and $\{\sigma_n\}$ be sequences of nonnegative real numbers such that*

$$a_{n+1} \leq (1 - \gamma_n) a_n + \sigma_n, \; n \geq 0,$$

where $\{\gamma_n\}$ is a sequence in $(0,1)$ such that;

(i) $\lim_{n\to\infty} \gamma_n = 0$ and $\sum \gamma_n = \infty$;

(ii) $\lim_{n\to\infty} \frac{\sigma_n}{\gamma_n} \leq 0$ or $\sum |\sigma_n| < \infty$.

Then $\lim_{n\to\infty} a_n = 0$.

3. Main Results

3.1. Parallel Algorithm (Weak Convergence Results)

Theorem 3.1. *Let $T : H_1 \to H_1$ and $U : H_2 \to H_2$ be k_1 and k_2-strictly pseudocontractive mappings with $k = \max\{k_1, k_2\}$. Let also $A : H_1 \to H_2$ be a bounded linear operator with its adjoint A^*. Assume that $\Gamma \neq \emptyset$ and let $\{x_n\}$ be the sequence defined by*

$$\begin{cases} x_0 \in H_1 \text{ is chosen arbitrarily,} \\ u_n = x_n + \gamma A^* \big(T - I\big) A x_n, \\ x_{n+1} = \alpha_n u_n + (1 - \alpha_n) U u_n, \forall n \geq 0, \end{cases} \tag{3.1}$$

where $k < \alpha_n < 1$ and $\gamma \in (0, \frac{1-k}{L})$ with $L = \|AA^*\|$. Then, $x_n \rightharpoonup x^* \in \Gamma$.

Proof. First, we verify that $\{x_n\}$ is a Fejer monotone sequence on Γ.

Now, let $x^* \in \Gamma$. By Equation (3.1), Lemma 2.4, and the fact that U and T are k-strictly pseudocontractive mappings, we have

$$\begin{aligned} \|x_{n+1} - x^*\|^2 &= \|\alpha_n u_n + (1-\alpha_n) U u_n - x^*\|^2 \\ &= \|\alpha_n (u_n - x^*) + (1-\alpha_n)(U u_n - x^*)\|^2 \\ &= \alpha_n \|u_n - x^*\|^2 + (1-\alpha_n)\|U u_n - x^*\|^2 - \alpha_n (1-\alpha_n) \|U u_n - u_n\|^2 \\ &\leq \alpha_n \|u_n - x^*\|^2 + (1-\alpha_n)\big(\|u_n - x^*\|^2 + k\|U u_n - u_n\|^2\big) \\ &\quad - \alpha_n (1-\alpha_n) \|U u_n - u_n\|^2 \\ &= \|u_n - x^*\|^2 - (\alpha_n - k)(1-\alpha_n) \|U u_n - u_n\|^2 , \end{aligned} \tag{3.2}$$

and

$$
\begin{aligned}
\|u_n - x^*\|^2 &= \left\|(x_n - x^*) + \gamma A^*\big(T - I\big)Ax_n\right\|^2 \\
&= \|x_n - x^*\|^2 + \gamma^2 \left\|A^*\big(T - I\big)Ax_n\right\|^2 + 2\gamma\left\langle x_n - x^*, A^*\big(T - I\big)Ax_n\right\rangle \\
&\leq \|x_n - x^*\|^2 + \gamma^2 \|AA^*\| \, \|(T - I)Ax_n\|^2 \\
&\quad + 2\gamma \left\langle Ax_n - Ax^*, TAx_n - Ax_n\right\rangle \\
&\leq \|x_n - x^*\|^2 + \gamma^2 L \|TAx_n - Ax_n\|^2 - \gamma(1 - k)\|TAx_n - Ax_n\|^2 \\
&= \|x_n - x^*\|^2 - \gamma(1 - k - \gamma L)\|TAx_n - Ax_n\|^2. \qquad (3.3)
\end{aligned}
$$

By Equations (3.2) and (3.3), we obtain that

$$
\begin{aligned}
\|x_{n+1} - x^*\|^2 \leq& \|x_n - x^*\|^2 - \gamma(1 - k - \gamma L)\|TAx_n - Ax_n\|^2 \\
&- (\alpha_n - k)(1 - \alpha_n) \|Uu_n - u_n\|^2 \qquad (3.4) \\
\leq& \|x_n - x^*\|^2 .
\end{aligned}
$$

Thus, $\{x_n\}$ is Fejer monotone. Therefore, the sequence $\{\|x_n - x^*\|\}$ converges. Next, we show that

$$\lim_{n\to\infty} \|TAx_n - Ax_n\| = 0 \;\text{ and }\; \lim_{n\to\infty} \|Uu_n - u_n\| = 0. \qquad (3.5)$$

This follows trivially from Equation (3.4) and the fact that $\{\|x_n - x^*\|\}$ converges. Finally, we show that $x_n \rightharpoonup x^*$. To show this, it suffices to show that $\omega_\omega \subset \Gamma$ (see Lemma 2.3). Let $x \in \omega_\omega(x_n)$, then there exists $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \rightharpoonup x$. By continuity of A, we have $Ax_{n_k} \rightharpoonup Ax$. By Equation (3.5) and demiclosedness of $(T - I)$ at zero, we deduce that $TAx = Ax$, this implies that $Ax \in Fix(T)$.

Similarly, by Equations (3.1), (3.5) and the fact that $x_{n_k} \rightharpoonup x$, we have $u_{n_k} \rightharpoonup x$. By (3.5) and demiclosedness of $(U - I)$ at zero, we have that $Ux = x$, this implies that $x \in Fix(U)$. Hence, $x \in \Gamma$. By uniqueness of limit and Lemma 2.3, we conclude that $x_n \rightharpoonup x^*$. ∎

Theorem 3.2. *Let $M \geq 1$ and $N \geq 1$ be integers, $T_r : H_1 \to H_1, 1 \leq r \leq M$ and $U_s : H_2 \to H_2, 1 \leq s \leq N$ be k_r and k_s-strictly pseudocontractive mappings with $0 \leq k_r < 1$ and $0 \leq k_s < 1$, respectively, and let $k = \max\{k_r \text{ and } k_s\}$, $A : H_1 \to H_2$ be a bounded linear operator with its adjoint A^*. Assume that $\Gamma \neq \emptyset$, and let $\{\beta_r\}_{r=1}^{M}$ and $\{\lambda_s\}_{s=1}^{N}$ be finite sequences of positive numbers such that $\sum_{r=1}^{M} \beta_r = 1$ and $\sum_{s=1}^{N} \lambda_s = 1$, respectively, and also let $\{x_n\}$ be the sequence defined by Algorithm (1.17), where $k < \alpha_n < 1$ for all n, and $\gamma \in (0, \frac{1-k}{L})$ with $L = \|AA^*\|$. Then, $x_n \rightharpoonup x^* \in \Gamma$.*

Proof. Let $U = \sum_{s=1}^{N} \lambda_s U_s$ and $T = \sum_{r=1}^{M} \beta_r T_r$. By Lemma 2.5, it follows that U and T are k-strictly pseudocontractive mappings, $Fix(U) = \bigcap_{s=1}^{N} Fix(U_s)$ and $Fix(T) = \bigcap_{r=1}^{M} Fix(T_r)$, respectively. Hence, we can rewrite Algorithm (1.17) as

$$
\begin{cases}
x_0 \in H_1 \text{ is chosen arbitrarily,} \\
u_n = x_n + \gamma A^*\big(T - I\big)Ax_n, \\
x_{n+1} = \alpha_n u_n + (1 - \alpha_n)Uu_n, \forall n \geq 0.
\end{cases} \qquad (3.6)
$$

Therefore, all the hypothesis of Theorem 3.1 are satisfied. Hence, the proof of this theorem follows directly from Theorem 3.1. ∎

In Theorem 3.2 Algorithm (1.17), the sequences $\{\beta_r\}_{r=1}^{M}$ and $\{\lambda_s\}_{s=1}^{N}$ are constants in the sense that both sequences are independent of n. In the next theorem, we consider a more general case by allowing those sequences to depend on n.

Theorem 3.3. *Let $M \geq 1$ and $N \geq 1$ be integers, $T_r : H_1 \to H_1, 1 \leq r \leq M$ and $U_s : H_2 \to H_2, 1 \leq s \leq N$ be k_r and k_s-strictly pseudocontractive mappings with $0 \leq k_r < 1$ and $0 \leq k_s < 1$, and let $k = \max\{k_r \text{ and } k_s\}$, $A : H_1 \to H_2$ be a bounded linear operator with its adjoint A^*. Assume that $\Gamma \neq \emptyset$, and let $\{x_n\}$ be the sequence defined by Algorithm (1.18). Where $\gamma \in (0, \frac{1-k}{L})$ with $L = \|AA^*\|$, $k < \alpha_n < 1$, $\{\lambda_r^n\}_{r=1}^{M}$ and $\{\beta_s^n\}_{s=1}^{N}$ are finite sequences of positive numbers satisfying the following conditions:*

(i) $\lim\limits_{n\to\infty} \alpha_n = 0, \sum \alpha_n = +\infty;$

(iii) $\sum_{n=0}^{\infty} \sum_{r=1}^{M} |\lambda_r^{n+1} - \lambda_r^n| < \infty$ and $\sum_{n=0}^{\infty} \sum_{s=1}^{N} |\beta_s^{n+1} - \beta_s^n| < \infty;$

(iii) $\sum_{r=1}^{M} \lambda_r^n = 1, \sum_{s=1}^{N} \beta_s^n = 1 \ \forall n, \inf\limits_{n\geq 1} \lambda_s^n > 0$ *and* $\inf\limits_{n\geq 1} \beta_r^n > 0.$

Then $x_n \rightharpoonup x^* \in \Gamma$.

Proof. **Step 1.** Here, we show that $\{x_n\}$ is Fejer monotone on Γ.

Write, for each $n \geq 1$, $U^n = \sum_{s=1}^{N} \lambda_s^n U_s$ and $T^n = \sum_{r=1}^{M} \beta_r^n T_r$. By Lemma 2.5, U^n and T^n are k-strictly pseudocontractive mappings, $Fix(U^n) = \bigcap_{s=1}^{N} Fix(U_s)$ and $Fix(T^n) = \bigcap_{r=1}^{M} Fix(T_r)$, respectively, and Algorithm (1.18) can be written as

$$\begin{cases} x_0 \in H_1 \text{ is chosen arbitrarily,} \\ u_n = x_n + \gamma A^* \big(T^n - I\big) A x_n, \\ x_{n+1} = \alpha_n u_n + (1-\alpha_n) U^n u_n, \forall n \geq 0. \end{cases} \tag{3.7}$$

Let $x^* \in \Gamma$. The following inequality is obtained the same way as Equation (3.4) in the proof of Theorem 3.1, replace U with U^n and T with T^n, we have

$$\begin{aligned} \|x_{n+1} - x^*\|^2 \leq \|x_n - x^*\|^2 - \gamma(1-k-\gamma L)\|T^n A x_n - A x_n\|^2 \\ - (\alpha_n - k)(1-\alpha_n) \|U^n u_n - u_n\|^2 . \end{aligned} \tag{3.8}$$

Thus, $\{x_n\}$ is Fejer monotone. Therefore, $\{\|x_n - x^*\|\}$ converges.

Step 2. Here, we show that

$$\lim_{n\to\infty} \|T^n A x_n - A x_n\| = 0 \ \text{ and } \lim_{n\to\infty} \|U^n u_n - u_n\| = 0. \tag{3.9}$$

By (3.7), we have that

$$\begin{aligned} &\left\|T^{n+1} A x_{n+1} - A x_{n+1} \right\|^2 = \|\alpha_n A u_n + (1-\alpha_n) U^n A u_n - T^{n+1} A x_{n+1} \|^2 \\ &= \left\|\alpha_n (A u_n - T^{n+1} A x_{n+1}) + (1-\alpha_n)(U^n A u_n - T^{n+1} A x_{n+1})\right\|^2 \\ &= \alpha_n \left\|A u_n - T^{n+1} A x_{n+1} \right\|^2 + (1-\alpha_n) \left\|U^n A u_n - T^{n+1} A x_{n+1} \right\|^2 \\ &\quad - \alpha_n (1-\alpha_n) \|U^n A u_n - A u_n\|^2 . \end{aligned} \tag{3.10}$$

Since T^n is strictly pseudocontractive, we deduce that

$$\begin{aligned} \|U^n A u_n - T^{n+1} A x_{n+1} \|^2 \leq \alpha_n^2 \|U^n A u_n - A u_n\|^2 \\ + k\|T^{n+1} A x_{n+1} - A x_{n+1} \|^2. \end{aligned} \tag{3.11}$$

From (3.10) and (3.11), we have

$$\left\|T^{n+1}Ax_{n+1}-Ax_{n+1}\right\|^2 \le k(1-\alpha_n)\|T^{n+1}Ax_{n+1}-Ax_{n+1}\|^2 + \alpha_n\|Au_n-T^{n+1}Ax_{n+1}\|^2 - \alpha_n(1-\alpha_n)^2\|U^nAu_n-Au_n\|^2. \quad (3.12)$$

Let $T^{n+1}Ax_{n+1}=T^nAx_{n+1}+z_n$, where $z_n:=\sum_{r=1}^{M}(\beta_r^{n+1}-\beta_r^n)T_rAx_{n+1}$, then

$$\|Au_n-T^{n+1}Ax_{n+1}\|^2 = \|Au_n-T^nAx_{n+1}\|^2 + 2\langle Au_n-T^nAx_{n+1},-z_n\rangle + \|z_n\|^2. \quad (3.13)$$

By Lemma 2.6 (ii), we deduce that

$$\begin{aligned}\|Au_n-T^nAx_{n+1}\|^2 &= \|Ax_n+A\gamma A^*\big(T^n-I\big)Ax_n-T^nAx_{n+1}\|^2\\ &= \|Ax_n-T^nAx_{n+1}\|^2+\|A\gamma A^*\big(T^n-I\big)Ax_n\|^2\\ &\quad +2\left\langle Ax_n-T^nAx_{n+1},A\gamma A^*\big(T^n-I\big)Ax_n\right\rangle\\ &\le \|Ax_n-T^nAx_{n+1}\|^2+\gamma^2L^2\|T^nAx_n-Ax_n\|^2\\ &\quad -\gamma(1-k)\|A^*(T^nAx_n-Ax_n)\|^2. \end{aligned} \quad (3.14)$$

From Equation (3.13), (3.14) and the fact that T^n is strictly pseudocontractive, we have

$$\begin{aligned}\|Au_n-T^{n+1}Ax_{n+1}\|^2 &\le \|Ax_n-Ax_{n+1}\|^2+k\|T^{n+1}Ax_{n+1}-Ax_{n+1}\|^2\\ &\quad +2k\left\langle T^{n+1}Ax_{n+1}-Ax_{n+1},-z_n\right\rangle\\ &\quad +(1+k)\|z_n\|^2+2\langle Au_n-T^nAx_{n+1},-z_n\rangle\\ &\quad +\gamma^2L^2\|T^nAx_n-Ax_n\|^2. \end{aligned} \quad (3.15)$$

From Equation (3.7) and the fact that U^n is strictly pseudocontractive, we have

$$\begin{aligned}\|Ax_n-Ax_{n+1}\|^2 &= \|\alpha_n(Au_n-Ax_n)+(1-\alpha_n)(U^nAu_n-Ax_n)\|^2\\ &= \alpha_n\|Au_n-Ax_n\|^2+(1-\alpha_n)\|U^nAu_n-Ax_n\|^2\\ &\quad -\alpha_n(1-\alpha_n)\|U^nAu_n-Au_n\|^2\\ &= \|Au_n-Ax_n\|^2+(1-\alpha_n)k\|U^nAu_n-Au_n\|^2\\ &\quad -\alpha_n(1-\alpha_n)\|U^nAu_n-Au_n\|^2\\ &\le \gamma^2L^2\|T^nAx_n-Ax_n\|^2\\ &\quad -(\alpha_n-k)(1-\alpha_n)\|U^nAu_n-Au_n\|^2. \end{aligned} \quad (3.16)$$

Thus, we deduce from Equations (3.12), (3.15) and (3.16) that

$$\begin{aligned}&(1-k)\left\|T^{n+1}Ax_{n+1}-Ax_{n+1}\right\|^2\\ &\quad\le 2\alpha_n\gamma^2L^2\|T^nAx_n-Ax_n\|^2+2\alpha_nk\left\langle T^{n+1}Ax_{n+1}-Ax_{n+1},-z_n\right\rangle\\ &\quad\ +(1+k)\alpha_n\|z_n\|^2+2\alpha_n\langle Au_n-T^nAx_{n+1},-z_n\rangle. \end{aligned} \quad (3.17)$$

Since $(1-k)>0$, we deduce from Equation (3.17) that

$$\begin{aligned}
&\left\|T^{n+1}Ax_{n+1}-Ax_{n+1}\right\|^2\\
&\quad\le\left(1-\alpha_n\gamma^2L^2\right)\|Ax_n-T^nAx_n\|^2+3\alpha_n\gamma^2L^2\|T^nAx_n-Ax_n\|^2\\
&\qquad+\frac{2\alpha_nk}{1-k}\left\langle T^{n+1}Ax_{n+1}-Ax_{n+1},-z_n\right\rangle\\
&\qquad+\frac{(1+k)}{1-k}\alpha_n\|z_n\|^2+\frac{2\alpha_n}{1-k}\left\langle Au_n-T^nAx_{n+1},-z_n\right\rangle
\end{aligned}\tag{3.18}$$

and this turn to implies that

$$\xi_{n+1}\le(1-\beta_n)\xi_n+\eta_n,\ \text{where}\tag{3.19}$$

$$\xi_n=\|T^nAx_n-Ax_n\|^2,\eta_n=\frac{\alpha_nM\|z_n\|}{1-k}+3\alpha_n\gamma^2L^2\|T^nAx_n-Ax_n\|^2\ \text{and}$$

$\beta_n=\alpha_n\gamma^2L^2$, where M is chosen appropriately such that

$$\begin{aligned}
2k|\left\langle T^{n+1}Ax_{n+1}-Ax_{n+1},-z_n\right\rangle|+2|\left\langle Au_n-T^nAx_{n+1},-z_n\right\rangle|\\
+(1+k)\|z_n\|\le M\|z_n\|.
\end{aligned}$$

Trivially, $\lim\limits_{n\to\infty}\beta_n=0$, $\sum\beta_n=\infty$ and $\lim\limits_{n\to\infty}\frac{\eta_n}{\beta_n}\le 0$. Hence, by Lemma 2.8, we deduce that

$$\lim_{n\to\infty}\|T^nAx_n-Ax_n\|=0.\tag{3.20}$$

On the other hand,

$$\begin{aligned}
\|U^{n+1}u_{n+1}-u_{n+1}\|^2&=\|U^{n+1}u_{n+1}-x_{n+1}\|^2+\|\gamma A^*(T^{n+1}-I)Ax_{n+1}\|^2\\
&\quad+2\left\langle U^{n+1}u_{n+1}-x_{n+1},-\gamma A^*(T^{n+1}-I)Ax_{n+1}\right\rangle.
\end{aligned}\tag{3.21}$$

Let $U^{n+1}u_{n+1}=U^nu_{n+1}+q_n$, where $q_n:=\sum_{s=1}^N(\lambda_s^{n+1}-\lambda_s^n)U_su_{n+1}$. The fact that U^n is k-strictly pseudocontractive mapping, we have that

$$\begin{aligned}
\|U^{n+1}u_{n+1}-x_{n+1}\|^2&=\|U^nu_{n+1}-x_{n+1}+q_n\|^2\\
&=\|U^nu_{n+1}-x_{n+1}\|^2+\|q_n\|^2+2\left\langle U^nu_{n+1}-x_{n+1},q_n\right\rangle\\
&\le\|u_{n+1}-x_{n+1}\|^2+k\|U^nu_{n+1}-u_{n+1}\|^2+\|q_n\|^2\\
&\quad+2\left\langle U^nu_{n+1}-x_{n+1},q_n\right\rangle\\
&=\|u_{n+1}-x_{n+1}\|^2+k\|U^{n+1}u_{n+1}-u_{n+1}\|^2\\
&\quad+(1+k)\|q_n\|^2+2\left\langle U^nu_{n+1}-x_{n+1},q_n\right\rangle\\
&\quad+2k\left\langle U^{n+1}u_{n+1}-u_{n+1},-q_n\right\rangle
\end{aligned}\tag{3.22}$$

From Equations (3.21) and (3.22), we have that

$$\begin{aligned}
&(1-k)\|U^{n+1}u_{n+1}-u_{n+1}\|^2\\
&\quad\le\|u_{n+1}-x_{n+1}\|^2+\|\gamma A^*(T^{n+1}-I)Ax_{n+1}\|^2\\
&\qquad+2\left\langle U^nu_{n+1}-x_{n+1},q_n\right\rangle+2k\left\langle U^{n+1}u_{n+1}-u_{n+1},-q_n\right\rangle\\
&\qquad+(1+k)\|q_n\|^2+2\left\langle U^{n+1}u_{n+1}-x_{n+1},-\gamma A^*(T^{n+1}-I)Ax_{n+1}\right\rangle.
\end{aligned}\tag{3.23}$$

On the other hand,

$$\|u_{n+1} - x_{n+1}\|^2 \leq \|u_{n+1} - u_n\|^2 - (1-\alpha_n)(\alpha_n - k)\|U^n u_n - u_n\|^2 \tag{3.24}$$

and

$$\begin{aligned}\|u_{n+1} - u_n\|^2 &= \|x_{n+1} - x_n\|^2 + \|\gamma A^*(T^{n+1} - I)Ax_{n+1} - \gamma A^*(T^n - I)Ax_n\|^2 \\ &\quad +2\left\langle x_{n+1} - x_n, \gamma A^*(T^{n+1} - I)Ax_{n+1} - \gamma A^*(T^n - I)Ax_n\right\rangle.\end{aligned} \tag{3.25}$$

Furthermore, we compute

$$\begin{aligned}\|x_{n+1} - x_n\|^2 &\leq \alpha_n \|u_n - x_n\|^2 + (1-\alpha_n)\|U^n u_n - u_n + u_n - x_n\|^2 \\ &\leq \|\gamma A^*(T^n - I)Ax_n\|^2 + (1-\alpha_n)\|U^n u_n - u_n\|^2 \\ &\quad + 2(1-\alpha_n)\langle U^n u_n - u_n, \gamma A^*(T^n - I)Ax_n\rangle.\end{aligned} \tag{3.26}$$

From Equations (3.25) and (3.26), we have

$$\begin{aligned}&\|u_{n+1} - u_n\|^2 \\ &\quad \leq \|\gamma A^*(T^{n+1} - I)Ax_{n+1} - \gamma A^*(T^n - I)Ax_n\|^2 \\ &\qquad +(1-\alpha_n)\|U^n u_n - u_n\|^2 + \|\gamma A^*(T^n - I)Ax_n\|^2 \\ &\qquad +2(1-\alpha_n)\langle U^n u_n - u_n, \gamma A^*(T^n - I)Ax_n\rangle \\ &\qquad +2\left\langle x_{n+1} - x_n, \gamma A^*(T^{n+1} - I)Ax_{n+1} - \gamma A^*(T^n - I)Ax_n\right\rangle.\end{aligned} \tag{3.27}$$

Thus, by (3.23), (3.24) and (3.27), we deduce that

$$\begin{aligned}&\|U^{n+1}u_{n+1} - u_{n+1}\|^2 \\ &\quad \leq (1-\alpha_n)\|U^n u_n - u_n\|^2 + \|\gamma A^*(T^n - I)Ax_n\|^2 \\ &\qquad +\|\gamma A^*(T^{n+1} - I)Ax_{n+1}\|^2 \\ &\qquad +\|\gamma A^*(T^{n+1} - I)Ax_{n+1} - \gamma A^*(T^n - I)Ax_n\|^2 \\ &\qquad +D\|q_n\| + \frac{2(1-\alpha_n)}{(1-k)}\langle U^n u_n - u_n, \gamma A^*(T^n - I)Ax_n\rangle \\ &\qquad +\frac{2}{(1-k)}\left\langle x_{n+1} - x_n, \gamma A^*(T^{n+1} - I)Ax_{n+1} - \gamma A^*(T^n - I)Ax_n\right\rangle \\ &\qquad +\frac{2}{(1-k)}\left\langle U^{n+1}u_{n+1} - x_{n+1}, -\gamma A^*(T^{n+1} - I)Ax_{n+1}\right\rangle,\end{aligned}$$

where D is chosen appropriately such that

$$\begin{aligned}\frac{2}{(1-k)}|\langle U^n u_{n+1} - x_{n+1}, q_n\rangle| + \frac{2k}{(1-k)}\left|\left\langle U^{n+1}u_{n+1} - u_{n+1}, -q_n\right\rangle\right| \\ + \frac{(1+k)}{(1-k)}\|q_n\| \leq D\|q_n\|.\end{aligned}$$

Thus, we deduce that

$$\xi_{n+1} \leq (1-\beta_n)\xi_n + \eta_n, \tag{3.28}$$

where

$$\begin{aligned}
\xi_n &= \|U^n u_n - u_n\|^2, \\
\beta_n &= \alpha_n, \text{ and} \\
\eta_n &= \|\gamma A^*(T^n - I)Ax_n\|^2 \\
&\quad + \|\gamma A^*(T^{n+1} - I)Ax_{n+1}\|^2 + \|\gamma A^*(T^{n+1} - I)Ax_{n+1} - \gamma A^*(T^n - I)Ax_n\|^2 \\
&\quad + D\|q_n\| + \frac{2(1-\alpha_n)}{(1-k)} \langle U^n u_n - u_n, \gamma A^*(T^n - I)Ax_n\rangle \\
&\quad + \frac{2}{(1-k)} \left\langle x_{n+1} - x_n, \gamma A^*(T^{n+1} - I)Ax_{n+1} - \gamma A^*(T^n - I)Ax_n \right\rangle \\
&\quad + \frac{2}{(1-k)} \left\langle U^{n+1} u_{n+1} - x_{n+1}, -\gamma A^*(T^{n+1} - I)Ax_{n+1} \right\rangle.
\end{aligned}$$

The fact that $\lim_{n\to\infty} \|T^n Ax_n - Ax_n\| = 0$, we deduce that $\lim_{n\to\infty} \beta_n = 0$, $\sum \beta_n = \infty$, and $\lim_{n\to\infty} \frac{\eta_n}{\beta_n} \leq 0$. Hence, by Lemma 2.8, we have that $\lim_{n\to\infty} \|U^n u_n - u_n\| = 0$, which complete the proof of step 2.

Step 3. Here, we show that

$$\lim_{n\to\infty} \|TAx_n - Ax_n\| = 0 \text{ and } \lim_{n\to\infty} \|Uu_n - u_n\| = 0. \tag{3.29}$$

From (1.17), we have

$$\begin{aligned}
\|x_{n+1} - x_n\| &= \|(1-\alpha_n)(U^n u_n - u_n) + u_n - x_n\| \\
&= \|(1-\alpha_n)(U^n u_n - u_n) + \gamma A^*(T^n - I)Ax_n\|.
\end{aligned}$$

In view of (3.9), we obtain that

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{3.30}$$

By (3.7), we have that

$$\begin{aligned}
\|u_{n+1} - u_n\| &= \|x_{n+1} - x_n\| \\
&\quad + \|\gamma A^*(T^{n+1} - I)A)x_{n+1} - \gamma A^*(T^n - I)A)x_n\|
\end{aligned}$$

Thus, in view of (3.9) and (3.30), we deduce that

$$\lim_{n\to\infty} \|u_{n+1} - u_n\| = 0. \tag{3.31}$$

Given Equations (3.9), (3.31) and the fact that U^n is $k-$strictly pseudocontractive, thus, for each n, U^n satisfies $\eta-$Lipschitzian condition, see Lemma 2.7, where η is chosen appropriately such that $\frac{1+k}{1-k} < \eta$. Hence, we have

$$\begin{aligned}
\|u_n - Uu_n\| &\leq \|u_n - U^n u_n\| + \|Uu_n - U^n u_n\| \\
&\leq \|u_n - U^n u_n\| + \eta \left\|u_n - U^{n-1}u_{n-1} + U^{n-1}u_{n-1} - U^{n-1}u_n\right\| \\
&\leq \|u_n - U^n u_n\| + \eta^2 \|u_n - u_{n-1}\| \\
&\quad + \eta \left\|u_n - u_{n-1} + u_{n-1} - U^{n-1}u_{n-1}\right\| \\
&\leq \|u_n - U^n u_n\| + \eta(\eta+1) \|u_n - u_{n-1}\| + \eta \left\|u_{n-1} - U^{n-1}u_{n-1}\right\|.
\end{aligned}$$

Thus, as $n \to \infty$, we have that

$$\lim_{n\to\infty} \|u_n - Uu_n\| = 0.$$

Similarly, from the fact that $\lim_{n\to\infty} \|Ax_n - T^n Ax_n\| = 0$, $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$, and T^n is $k-$strictly pseudocontractive, we also obtain that $\lim_{n\to\infty} \|Ax_n - TAx_n\| = 0$.

Step 4. Here, we show that $x_n \rightharpoonup x^*$. To show this, it suffices to show that $\omega_\omega \subseteq \Gamma$, see Lemma 2.3. Now, let $x \in \omega_\omega(x_n)$, this implies that there exists a sub-sequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \rightharpoonup x$. By continuity of A, we have that $Ax_{n_k} \rightharpoonup Ax$. By (3.29) and demiclosed property of $(T - I)$ at zero, we deduce that $TAx = Ax$, this implies that $Ax \in Fix(T)$.

Similarly, by (3.1), (3.5) and the fact that $x_{n_k} \rightharpoonup x$, it follows that $u_{n_k} \rightharpoonup x$. Also by (3.29) and demiclosed property of $(U - I)$ at zero, we also deduce that $Ux = x$. This implies that $x \in Fix(U)$. Hence, $x \in \Gamma$. By uniqueness of limit and Lemma 2.3, we conclude that $x_n \rightharpoonup x^*$. ■

3.2. Strong Convergence Theorems

It is known that, in finite-dimensional Hilbert space, weak and strong convergence are equivalent, and it is different from infinite dimensional space where they are not equivalent. For the example of weak convergence which is not strong convergence, see [2] and reference therein. In Theorems 3.1, 3.2, and 3.3, Algorithm (3.1), (1.17), and (1.18) converges weakly in an infinite dimensional Hilbert space, to get strong convergence results, we modify these algorithms.

The following theorem is an extension of Theorem 3.2 from a weak convergence to a strong convergence.

Theorem 3.4. *Let C_1 be a nonempty closed convex subset of H_1, $M \geq 1$ and $N \geq 1$ be integers, $T_r : C_1 \to H_1, 1 \leq r \leq M$, and $U_s : H_2 \to H_2, 1 \leq s \leq N$ be k_r and k_s- strictly pseudocontractive mappings with $0 \leq k_r < 1$ and $0 \leq k_s < 1$, respectively. And also let $k = \max\{k_r \text{ and } k_s, \text{ for } r = 1, 2, 3, ..., M \text{ and } s = 1, 2, 3, ..., N\}$ such that for all n, $k < \alpha_n < 1$, and $A : H_1 \to H_2$ be a bounded linear operator with its adjoint A^*. Assume that $\Gamma \neq \emptyset$, and let $\{\lambda_s\}_{r=1}^N$ and $\{\beta_r\}_{r=1}^M$ be finite sequences of positive numbers such that $\sum_{s=1}^N \lambda_s = 1$ and $\sum_{r=1}^M \beta_r = 1$, and also let $\{x_n\}$ be the sequence generated by*

$$\begin{cases} x_1 \in C_1, \\ u_n = x_n + \gamma A^*\big(\sum_{r=1}^M \beta_r T_r - I\big)Ax_n, \\ y_n = \alpha_n u_n + (1-\alpha_n)\sum_{s=1}^N \lambda_s U_s u_n, \\ C_{n+1} = \Big\{z \in C_n : \|y_n - z\|^2 \leq \|u_n - z\|^2 \leq \|x_n - z\|^2\Big\}, \\ x_{n+1} = P_{C_{n+1}}(x_1), \forall n \geq 1, \end{cases} \tag{3.32}$$

where $\gamma \in (0, \frac{1-k}{L})$ with $L = \|AA^*\|$. Then $x_n \to x^* \in \Gamma$.

Proof. Let $U = \sum_{s=1}^N \lambda_s U_s$ and $T = \sum_{r=1}^M \beta_r T_r$. By Lemma 2.5, U and T are $k-$ strictly pseudocontractive, $Fix(U) = \bigcap_{s=1}^N Fix(U_s)$ and $Fix(T) = \bigcap_{r=1}^M Fix(T_r)$, respectively.

Therefore, Algorithm (3.32) can be rewrite as

$$\begin{cases} x_1 \in C_1, \\ u_n = x_n + \gamma A^*\big(T - I\big)Ax_n, \\ y_n = \alpha_n u_n + (1-\alpha_n)Uu_n, \\ C_{n+1} \ = \Big\{z \in C_n : \|y_n - z\|^2 \leq \|u_n - z\|^2 \leq \|x_n - z\|^2\Big\}, \\ x_{n+1} \ = P_{C_{n+1}}(x_1), \forall n \geq 1. \end{cases} \tag{3.33}$$

Step 1. Here, we show that for each $n \geq 1$, C_n is closed and convex. Trivially, C_n is closed. Next, we show that C_n is convex. To show this, it suffices to show that for each $r_1, r_2 \in C_n$, and $\xi \in (0,1)$, $\xi r_1 + (1-\xi)r_2 \in C_n$. Now, we compute

$$\begin{aligned} \|y_n - \xi r_1 - (1-\xi)r_2\|^2 &= \xi\|y_n - r_1\|^2 + (1-\xi)\|y_n - r_2\|^2 - (1-\xi)\xi\|r_1 - r_2\|^2 \\ &\leq \xi\|u_n - r_1\|^2 + (1-\xi)\|u_n - r_2\|^2 - (1-\xi)\xi\|r_1 - r_2\|^2 \\ &= \|u_n - \xi r_1 - (1-\xi)r_2\|^2. \end{aligned}$$

Similarly, we also obtain that

$$\|y_n - \xi r_1 - (1-\xi)r_2\|^2 \leq \|x_n - \xi r_1 - (1-\xi)r_2\|^2.$$

This show that for each $r_1, r_2 \in C_n$, $\xi r_1 + (1-\xi)r_2 \in C_n$.

Step 2. Here, we show that $\Gamma \subset C_n$. Following the proof of Equation (3.4), for each $q \in \Gamma$, it follows from (3.33) that

$$\begin{aligned} \|y_n - q\|^2 \leq \|x_n - q\|^2 &- \gamma(1 - k - \gamma L)\|TAx_n - Ax_n\|^2 \\ &- (\alpha_n - k)(1-\alpha_n)\left\|Uu_n - u_n\right\|^2. \end{aligned} \tag{3.34}$$

Thus, $\Gamma \in C_n$.

Step 3. Here we show that $\{x_n\}$ is a Cauchy sequence. By the definition of C_{n+1} , we deduce that $x_n = P_{C_n}(x_1)$. The fact that $\Gamma \subset C_{n+1} \subset C_n$ and $x_{n+1} = P_{C_{n+1}}(x_1) \in C_n$, it follows that

$$\|x_{n+1} - x_1\| \leq \|x_1 - q\|, \text{ for all } n \in \mathbb{N} \text{ and } q \in \Gamma.$$

Thus, $\{x_n\}$ is bounded. By Lemma 2.7, we obtain that

$$\begin{aligned} \|x_{n+1} - x_n\|^2 + \|x_n - x_1\|^2 &= \|x_{n+1} - P_{C_n}x_1\|^2 + \|x_1 - P_{C_n}x_1\|^2 \\ &\leq \|x_{n+1} - x_1\|^2. \end{aligned}$$

It is clear that $\{\|x_n - x_1\|\}$ is a monotone increasing sequence. By the boundedness of $\{x_n\}$, we deduce that the

$$\lim_{n\to\infty} \|x_n - x_1\| \text{ exist.} \tag{3.35}$$

Let $m, n \in \mathbb{N}$ with $m > n$, by the definition of C_n, we have $x_m = P_{C_m}(x_1) \in C_n$. It follows from Lemma (2.7) that

$$\begin{aligned} \|x_m - x_n\|^2 + \|x_n - x_1\|^2 &= \|x_m - P_{C_n}x_1\|^2 + \|x_1 - P_{C_n}x_1\|^2 \\ &\leq \|x_m - x_1\|^2. \end{aligned} \tag{3.36}$$

By Equations (3.35) and (3.36), we have that $\lim_{n\to\infty} \|x_m - x_n\| = 0$. Therefore, $\{x_n\}$ is a Cauchy sequence.

Step 4. Here, we show that

$$\lim_{n\to\infty} \|TAx_n - Ax_n\| = 0 \text{ and } \lim_{n\to\infty} \|Uu_n - u_n\| = 0. \tag{3.37}$$

From (3.33) and the fact that $x_{n+1} \in C_n$, we have

$$\begin{aligned} \|y_n - x_{n+1}\|^2 \leq \|x_n - x_{n+1}\|^2 - \gamma(1-k-\gamma L)\|TAx_n - Ax_n\|^2 \\ - (\alpha_n - k)(1-\alpha_n)\|Uu_n - u_n\|^2 . \end{aligned} \tag{3.38}$$

From (3.38) and the fact that $\{x_n\}$ is a Cauchy sequence, we deduce that

$$\lim_{n\to\infty} \|y_n - x_{n+1}\| = 0. \tag{3.39}$$

Equation (3.37) follows from Equations (3.38), (3.39), and the fact that $\{x_n\}$ is a Cauchy sequence. On the other hand,

$$\begin{aligned} \|u_n - x_n\| &\leq \|u_n - x_{n+1}\| + \|x_{n+1} - x_n\| \\ &\leq 2\|x_{n+1} - x_n\|. \end{aligned} \tag{3.40}$$

From (3.40) and the fact that $\{x_n\}$ is a Cauchy sequence, we deduce that

$$\lim_{n\to\infty} \|u_n - x_n\| = 0. \tag{3.41}$$

Similarly, from Equation (3.39) and the fact that $\{x_n\}$ is a Cauchy sequence, we deduce that

$$\lim_{n\to\infty} \|u_n - y_n\| = 0. \tag{3.42}$$

From Equation (3.34), we deduce that

$$\begin{aligned} \|TAx_n - Ax_n\|^2 &\leq \frac{\|x_n - q\|^2 - \|y_n - q\|^2}{\gamma(1-k-\gamma L)} \\ &\leq \frac{\|y_n - x_n\|\left[\,\|x_n - y_n\| + 2\,\|y_n - q\|\,\right]}{\gamma(1-k-\gamma L)}. \end{aligned} \tag{3.43}$$

Therefore, by (3.39), we have that $\lim_{n\to\infty} \|TAx_n - Ax_n\| = 0$. Similarly, by (3.41), we have that $\lim_{n\to\infty} \|Uu_n - u_n\| = 0$.

Step 5. Finally, we show that $x_n \to x^*$. Since $\{x_n\}$ is Cauchy, we assume that $x_n \to x^*$. Thus, by (3.33), we have that $u_n \to x^*$. This implies that $u_n \rightharpoonup x^*$. The fact that $\lim_{n\to\infty} \|Uu_n - u_n\| = 0$ together with demiclosedness of $(U - I)$ at zero, we deduce that $x^* \in Fix(U)$.

On the other hand, by the definition of A and $x_n \to x^*$, we have that $Ax_n \to Ax^*$. This implies that $Ax_n \rightharpoonup x^*$. The fact that $\lim_{n\to\infty} \|TAx_n - Ax_n\| = 0$ together with the demiclosedness of $(T - I)$ at zero, we deduce that $Ax^* \in Fix(T)$. Hence, $x^* \in \Gamma$, this turn to implies that $x_n \to x^*$. ■

In Theorem 3.3, the sequences $\{\lambda_s\}_{r=1}^N$ and $\{\beta_r\}_{r=1}^M$ are constant in the sense that both are independent on n. In the next theorem, we considered a more general case by allowing those sequences to depend on n. Theorem 3.5 does not only extends Theorem 3.4 but also extends and improves Theorem 3.2 from a weak convergence to a strong convergence.

Theorem 3.5. *Let C_1 be a nonempty closed convex subset of H_1, $M \geq 1$ and $N \geq 1$ be integers, $T_r : C_1 \to H_1, 1 \leq r \leq M$ and $U_s : H_2 \to H_2, 1 \leq s \leq N$ be k_r and k_s-strictly pseudocontractive mappings with $0 \leq k_r < 1$ and $0 \leq k_s < 1$ and let $k = \max\{k_r \text{ and } k_s, \text{ for } 1 \leq r \leq M \text{ and } 1 \leq s \leq N\}$, $A : H_1 \to H_2$ be a bounded linear operator with its adjoint A^*. Assume that $\Gamma \neq \emptyset$, and let $\{x_n\}$ be the sequence defined by*

$$\begin{cases} x_1 \in C_1, \\ y_n = \alpha_n u_n + (1-\alpha_n) \sum_{s=1}^{N} \lambda_s^n U_s u_n, \\ u_n = x_n + \gamma A^* \big(\sum_{r=1}^{M} \beta_r^n T_r - I \big) A x_n, \\ C_{n+1} = \Big\{ z \in C_n : \|y_n - z\|^2 \leq \|u_n - z\|^2 \leq \|x_n - z\|^2 \Big\}, \\ x_{n+1} = P_{C_{n+1}}(x_1), \forall n \geq 1, \end{cases} \tag{3.44}$$

where $\gamma \in (0, \frac{1-k}{L})$ with $L = \|AA^*\|$, $k < \{\alpha_n\} < 1$, $\{\lambda_r^n\}_{r=1}^{M}$ and $\{\beta_s^n\}_{s=1}^{N}$ are finite sequences of positive numbers satisfies the following conditions:

(i) $\lim_{n\to\infty} \alpha_n = 0, \sum \alpha_n = +\infty;$

(ii) $\sum_{n=0}^{\infty} \sum_{s=1}^{N} |\lambda_s^{n+1} - \lambda_s^n| < \infty$ and $\sum_{n=0}^{\infty} \sum_{r=1}^{M} |\beta_r^{n+1} - \beta_r^n| < \infty;$

(iii) $\sum_{s=1}^{N} \lambda_s^n = 1, \sum_{r=1}^{M} \beta_r^n = 1 \ \forall n, \inf_{n\geq 1} \lambda_s^n > 0$ *and* $\inf_{n\geq 1} \beta_r^n > 0.$

Then, $x_n \to x^* \in \Gamma$.

Proof. Write for each $n \geq 1$, $U^n = \sum_{s=1}^{N} \lambda_s^n U_s$ and $T^n = \sum_{r=1}^{M} \beta_r^n T_r$. By Lemma 2.5, U^n and T^n are $k-$strictly pseudocontractive, $Fix(U^n) = \bigcap_{s=1}^{N} Fix(U_s)$ and $Fix(T^n) = \bigcap_{r=1}^{M} Fix(T_r)$, respectively. Therefore, we can rewrite Algorithm (3.44) as

$$\begin{cases} x_1 \in C_1, \\ y_n = \alpha_n u_n + (1-\alpha_n) U^n u_n, \\ u_n = x_n + \gamma A^* \big(T^n - I \big) A x_n, \\ C_{n+1} = \Big\{ z \in C_n : \|y_n - z\|^2 \leq \|u_n - z\|^2 \leq \|x_n - z\|^2 \Big\}, \\ x_{n+1} = P_{C_{n+1}}(x_1), \forall n \geq 1. \end{cases} \tag{3.45}$$

By Theorem 3.4, we see that C_n is closed and convex, $\Gamma \subset C_n$, and $\{x_n\}$ is Cauchy. Next, we show that

$$\lim_{n\to\infty} \|T^n A x_n - A x_n\| = 0 \text{ and } \lim_{n\to\infty} \|U^n u_n - u_n\| = 0. \tag{3.46}$$

From (3.45) and the fact that $x_{n+1} \in C_n$. Following the same steps as in the proof of Theorem 3.1, in particular Equation (3.4), we obtain that

$$\begin{aligned} \|y_n - x_{n+1}\|^2 \leq \|x_n - x_{n+1}\|^2 - \gamma(1-k-\gamma L)\|T^n A x_n - A x_n\|^2 \\ - (\alpha_n - k)(1-\alpha_n) \|U^n u_n - u_n\|^2 . \end{aligned} \tag{3.47}$$

The fact that $\{x_n\}$ is Cauchy, we obtain that

$$\lim_{n\to\infty} \|y_n - x_{n+1}\| = 0. \tag{3.48}$$

From (3.48) and the fact that $\{x_n\}$ is Cauchy, we deduce from (3.47) that

$$\limsup_{n\to\infty} \|T^n A x_n - A x_n\| = 0 \text{ and } \limsup_{n\to\infty} \|U^n u_n - u_n\| = 0. \tag{3.49}$$

Next, we show that the limit of (3.49) actually exists.

From (3.45), we deduce that

$$\begin{aligned}\|T^{n+1}Ax_{n+1} - Ax_{n+1}\|^2 &= \|T^{n+1}Ax_{n+1} - Ay_n\|^2 + \|Ay_n - Ax_{n+1}\|^2 \\ &\quad + 2\left\langle T^{n+1}Ax_{n+1} - Ay_n, Ay_n - Ax_{n+1}\right\rangle \\ &= \alpha_n\|Au_n - T^{n+1}Ax_{n+1}\|^2 \\ &\quad + (1-\alpha_n)\|U^nAu_n - T^{n+1}Ax_{n+1}\|^2 \\ &\quad - \alpha_n(1-\alpha_n)\|U^nAu_n - Au_n\|^2 + \|Ay_n - Ax_{n+1}\|^2 \\ &\quad + 2\left\langle T^{n+1}Ax_{n+1} - Ay_n, Ay_n - Ax_{n+1}\right\rangle. \end{aligned} \tag{3.50}$$

Since each U^n and T^n are k-strictly pseudocontractive, we obtain that

$$\begin{aligned}\|U^nAu_n - T^{n+1}Ax_{n+1}\|^2 &\le \|U^nAu_n - Ax_{n+1}\|^2 + k\|T^{n+1}Ax_{n+1} - Ax_{n+1}\|^2 \\ &\le \|Au_n - Ax_{n+1}\|^2 + k\|U^nAu_n - Au_n\|^2 \\ &\quad + k\|T^{n+1}Ax_{n+1} - Ax_{n+1}\|^2 \\ &\le \|Ax_n - Ax_{n+1}\|^2 + \|A\gamma A^*(T^n - I)Ax_n\|^2 \\ &\quad + 2\left\langle Ax_n - Ax_{n+1}, A\gamma A^*(T^n - I)Ax_n\right\rangle \\ &\quad + k\|U^nAu_n - Au_n\|^2 + k\|T^{n+1}Ax_{n+1} - Ax_{n+1}\|^2. \end{aligned} \tag{3.51}$$

Let $T^{n+1}Ax_{n+1} = T^nAx_{n+1} + z_n$, where $z_n := \sum_{r=1}^{M}(\beta_r^{n+1} - \beta_r^n)T_rAx_{n+1}$. The fact that T^n is k-strictly pseudocontractive, we have

$$\begin{aligned}\|Au_n - T^{n+1}Ax_{n+1}\|^2 &= \|Au_n - T^nAx_{n+1}\|^2 + \|z_n\|^2 + 2\left\langle Au_n - T^nAx_{n+1}, -z_n\right\rangle \\ &\le \|Au_n - Ax_{n+1}\|^2 + k\|T^nAx_{n+1} - Ax_{n+1}\|^2 + \|z_n\|^2 \\ &\quad + 2\left\langle Au_n - T^nAx_{n+1}, -z_n\right\rangle \\ &\le \|Ax_n - Ax_{n+1}\|^2 + \|A\gamma A^*(T^n - I)Ax_n\|^2 \\ &\quad + 2\left\langle Ax_n - Ax_{n+1}, A\gamma A^*(T^n - I)Ax_n\right\rangle \\ &\quad + k\|Ax_{n+1} - T^{n+1}Ax_{n+1}\|^2 \\ &\quad + (1+k)\|z_n\|^2 + 2k\left\langle T^{n+1}Ax_{n+1} - Ax_{n+1}, -z_n\right\rangle \\ &\quad + 2\left\langle Au_n - T^nAx_{n+1}, -z_n\right\rangle. \end{aligned} \tag{3.52}$$

By substituting Equations (3.51) and (3.52) into (3.50), we have

$$\begin{aligned}(1-k)\left\|T^{n+1}Ax_{n+1} - Ax_{n+1}\right\|^2 &\le \|Ax_n - Ax_{n+1}\|^2 + \gamma^2L\|T^nAx_n - Ax_n\|^2 \\ &\quad + 2\left\langle Ax_n - Ax_{n+1}, A\gamma A^*(T^n - I)Ax_n\right\rangle + (1+k)\alpha_n\|z_n\|^2 \\ &\quad + 2\alpha_nk\left\langle T^{n+1}Ax_{n+1} - Ax_{n+1}, -z_n\right\rangle + 2\alpha_n\left\langle Au_n - T^nAx_{n+1}, -z_n\right\rangle \\ &\quad + \|Ay_n - Ax_{n+1}\|^2 + 2\left\langle T^{n+1}Ax_{n+1} - Ay_n, Ay_n - Ax_{n+1}\right\rangle. \end{aligned}$$

Thus, the fact that $(1-k)>0$ and noticing that $\gamma^2 L<(1-k)^2$, we deduce that

$$\begin{aligned}\left\|T^{n+1} Ax_{n+1} - Ax_{n+1}\right\|^2 \leq{}& \|Ax_n - Ax_{n+1}\|^2 + (1-\alpha_n)\|T^n Ax_n - Ax_n\|^2 \\ &+ \alpha_n \|T^n Ax_n - Ax_n\|^2 \\ &+ \frac{1}{1-k}\langle Ax_n - Ax_{n+1}, A\gamma A^*(T^n - I)Ax_n\rangle \\ &+ (1+k)\alpha_n\|z_n\|^2 + \frac{2\alpha_n k}{1-k}\left\langle T^{n+1}Ax_{n+1} - Ax_{n+1}, -z_n\right\rangle \\ &+ \frac{2\alpha_n}{1-k}\langle Au_n - T^n Ax_{n+1}, -z_n\rangle + \|Ay_n - Ax_{n+1}\|^2 \\ &+ \frac{2}{1-k}\left\langle T^{n+1}Ax_{n+1} - Ay_n, Ay_n - Ax_{n+1}\right\rangle. \qquad (3.53)\end{aligned}$$

This implies that

$$\xi_{n+1} \leq (1-\beta_n)\xi_n + \eta_n,$$

where

$$\begin{aligned}\xi_n &= \|T^n Ax_n - Ax_n\|^2, \\ \beta_n &= \alpha_n \text{ and} \\ \eta_n &= \alpha_n M\|z_n\| + \alpha_n \|T^n Ax_n - Ax_n\|^2 + \|Ay_n - Ax_{n+1}\|^2 \\ &\quad + \|Ax_n - Ax_{n+1}\|^2 + \frac{2}{1-k}\left\langle T^{n+1}Ax_{n+1} - Ay_n, Ay_n - Ax_{n+1}\right\rangle \\ &\quad + \frac{1}{1-k}\langle Ax_n - Ax_{n+1}, A\gamma A^*(T^n - I)Ax_n\rangle.\end{aligned}$$

where M is chosen appropriately such that

$$\begin{aligned}\frac{2k}{1-k}\left|\left\langle T^{n+1}Ax_{n+1} - Ax_{n+1}, -z_n\right\rangle\right| + \frac{2}{1-k}\left|\langle Au_n - T^n Ax_{n+1}, -z_n\rangle\right| \\ + (1+k)\|z_n\| \leq M\|z_n\|.\end{aligned}$$

Clearly, $\lim_{n\to\infty}\beta_n = 0, \sum\beta_n = \infty$, and $\lim_{n\to\infty}\frac{\eta_n}{\beta_n} \leq 0$. Thus, by Lemma 2.8, we deduce that

$$\lim_{n\to\infty}\|T^n Ax_n - Ax_n\| = 0. \qquad (3.54)$$

On the other hand,

$$\begin{aligned}\|U^{n+1}u_{n+1} - u_{n+1}\|^2 = {}& \|U^{n+1}u_{n+1} - y_n\|^2 + \|u_{n+1} - y_n\|^2 \\ &+ 2\left\langle U^{n+1}u_{n+1} - y_n, y_n - u_{n+1}\right\rangle. \qquad (3.55)\end{aligned}$$

Let $U^{n+1}u_{n+1} = U^n u_{n+1} + w_n$, where $w_n = \sum_{s=1}^{N}(\lambda_s^{n+1} - \lambda_s^n)U_s u_{n+1}$. The fact that each U^n is k-strictly pseudocontractive mapping, it follows that

$$\begin{aligned}\|U^{n+1}u_{n+1} - y_n\|^2 &= \|U^n u_{n+1} - y_n\|^2 + 2\langle U^n u_{n+1} - y_n, w_n\rangle + \|w_n\|^2 \\ &\leq \|u_{n+1} - y_n\|^2 + k\|U^n u_{n+1} - u_{n+1}\|^2 \\ &\quad + 2\langle U^n u_{n+1} - y_n, w_n\rangle + \|w_n\|^2 \\ &\leq \|u_{n+1} - y_n\|^2 + k\|U^{n+1}u_{n+1} - u_{n+1}\|^2 + (1+k)\|w_n\|^2 \\ &\quad + 2\langle U^n u_{n+1} - y_n, w_n\rangle + 2k\left\langle U^{n+1}u_{n+1} - u_{n+1}, -w_n\right\rangle. \qquad (3.56)\end{aligned}$$

On the other hand,

$$\|u_{n+1} - y_n\|^2 \leq \|u_{n+1} - u_n\|^2 - (1-\alpha_n)(\alpha_n - k)\|U^n u_n - u_n\|^2 \tag{3.57}$$

and

$$\begin{aligned}\|u_{n+1} - u_n\|^2 = \|x_{n+1} - x_n\|^2 + \|\gamma A^*(T^{n+1} - I)Ax_{n+1} - \gamma A^*(T^n - I)Ax_n\|^2 \\ + 2\left\langle x_{n+1} - x_n, \gamma A^*(T^{n+1} - I)Ax_{n+1} - \gamma A^*(T^n - I)Ax_n\right\rangle.\end{aligned} \tag{3.58}$$

From (3.54) and the fact that $\{x_n\}$ is Cauchy, it follows from (3.58) that

$$\lim_{n\to\infty} \|u_{n+1} - u_n\| = 0. \tag{3.59}$$

From (3.56) and (3.57), we have

$$\begin{aligned}\|U^{n+1} u_{n+1} - y_n\|^2 \leq k\|U^{n+1} u_{n+1} - u_{n+1}\|^2 + \|u_{n+1} - u_n\|^2 \\ - (1-\alpha_n)(\alpha_n - k)\|U^n u_n - u_n\|^2 + (1+k)\|w_n\|^2 \\ + 2\left\langle U^n u_{n+1} - y_n, w_n\right\rangle + 2k\left\langle U^{n+1} u_{n+1} - u_{n+1}, -w_n\right\rangle.\end{aligned} \tag{3.60}$$

By substituting (3.60) into (3.55) and the fact that $k < 1$, we have

$$\begin{aligned}(1-k)\|U^{n+1} u_{n+1} - u_{n+1}\|^2 \leq \|u_{n+1} - u_n\|^2 + (1-\alpha_n)\|U^n u_n - u_n\|^2 \\ + (1+k)\|w_n\|^2 + \|u_{n+1} - y_n\|^2 \\ + 2\left\langle U^n u_{n+1} - y_n, w_n\right\rangle \\ + 2k\left\langle U^{n+1} u_{n+1} - u_{n+1}, -w_n\right\rangle \\ + 2\left\langle U^{n+1} u_{n+1} - y_n, y_n - u_{n+1}\right\rangle.\end{aligned} \tag{3.61}$$

By (3.61) and the fact that $(1-k) > 0$, we have that

$$\begin{aligned}&\|U^{n+1} u_{n+1} - u_{n+1}\|^2 \\ &\leq (1-\alpha_n)\|U^n u_n - u_n\|^2 + \|u_{n+1} - u_n\|^2 \\ &\quad +(1+k)\|w_n\|^2 + \|u_{n+1} - y_n\|^2 \\ &\quad +\frac{2}{1-k}\Big(\left\langle U^n u_{n+1} - y_n, w_n\right\rangle + 2k\left\langle U^{n+1} u_{n+1} - u_{n+1}, -w_n\right\rangle\Big) \\ &\quad +2\|U^{n+1} u_{n+1} - y_n\|\|y_n - u_{n+1}\|.\end{aligned} \tag{3.62}$$

This implies that

$$\xi_{n+1} \leq (1-\beta_n)\xi_n + \eta_n,$$

where

$$\xi_n = \|U^n u_n - u_n\|^2,$$
$$\beta_n = \alpha_n \text{ and}$$
$$\eta_n = M_1\|w_n\| + \|u_{n+1} - u_n\|^2 + M_2\|u_{n+1} - u_n\|,$$

where M_1 and M_2 are chosen appropriately such that

$$\begin{aligned}(1+k)\|w_n\| + \frac{2}{1-k}\left|\left\langle U^n u_{n+1} - y_n, w_n\right\rangle\right| + \frac{2k}{1-k}\left|\left\langle U^{n+1} u_{n+1} - u_{n+1}, -w_n\right\rangle\right| \\ \leq M_1\|w_n\|\end{aligned}$$

and

$$2\|U^{n+1} u_{n+1} - y_n\|\|y_n - u_{n+1}\| \leq M_2\|y_n - u_{n+1}\|.$$

From (3.48) and (3.54), we have that

$$\lim_{n\to\infty} \|y_n - u_{n+1}\| = 0.$$

Clearly, that $\lim_{n\to\infty} \beta_n = \infty, \sum \beta_n = 0$ and $\lim_{n\to\infty} \frac{\eta_n}{\beta_n} \leq 0$. Thus, by Lemma (2.8), we deduce that

$$\lim_{n\to\infty} \|U^n u_n - u_n\| = 0.$$

This completes the proof of Equation (3.46). Following the same steps as in the proof of Equation (3.29) and together with Equation (3.46), we deduce that

$$\lim_{n\to\infty} \|TAx_n - Ax_n\| = 0 \text{ and } \lim_{n\to\infty} \|Uu_n - u_n\| = 0. \tag{3.63}$$

Finally, we show that $x_n \to x^*$. Since $\{x_n\}$ is a Cauchy sequence, we assume that $x_n \to x^*$. By (3.45) and the fact that the $\lim_{n\to\infty} \|TAx_n - Ax_n\| = 0$, we have that $u_n \to x^*$, this implies that $u_n \rightharpoonup x^*$. The fact that $\lim_{n\to\infty} \|Uu_n - u_n\| = 0$ together with the demiclosedness of $(U - I)$ at zero, we deduce that $x^* \in Fix(U)$.

On the other hand, by the definition of A and $x_n \to x^*$, we have that $Ax_n \to Ax^*$, this implies that $Ax_n \rightharpoonup x^*$. The fact that $\lim_{n\to\infty} \|TAx_n - Ax_n\| = 0$ together with the demiclosedness of $(T - I)$ at zero, we deduce that $Ax^* \in Fix(T)$. Hence, $x^* \in \Gamma$, this show that $x_n \to x^*$. This completes the proof. ∎

3.3. Cyclic Algorithm

Theorem 3.6. *Let $T_r : H_1 \to H_1$, and $U_r : H_2 \to H_2$ for $r = 1, 2, 3, ..., M$ be k_r−strictly pseudocontractive mappings with $0 \leq k_r < 1$, and let $k = \max\{k_r, 1 \leq r \leq M\}$, $A : H_1 \to H_2$ be a bounded linear operator with its adjoint A^*. Assume that, for $N = M$, the solution set Γ of Problem (1.20) is nonempty, and let $\{x_n\}$ be the sequence generated by*

$$\begin{cases} x_1 \in H_1; \\ u_n = x_n + \gamma A^*\big(T_{[n]} - I\big)Ax_n, \\ x_{n+1} = \alpha_n u_n + (1 - \alpha_n)U_{[n]}u_n, \forall n \geq 1, \end{cases} \tag{3.64}$$

where $[n] = n(\mathrm{mod}\, M)$ with mod function taking values in $\{1, 2, 3, ..., M\}$, $k < \alpha_n < 1$ and $\gamma \in (0, \frac{1-k}{L})$ with $L = \|AA^*\|$. Then $x_n \rightharpoonup x^* \in \Gamma$.

Proof. First, we show that $\{x_n\}$ is Fejer monotone with respect to Γ. Let $x^* \in \Gamma$, following the same steps as in the proof of Theorem 3.1, in particular Equation (3.4), we obtain that

$$\begin{aligned} \|x_{n+1} - x^*\|^2 \leq \|x_n - x^*\|^2 - \gamma(1 - k - \gamma L)\|T_{[n]}Ax_n - Ax_n\|^2 \\ - (\alpha_n - k)(1 - \alpha_n)\left\|U_{[n]}u_n - u_n\right\|^2. \end{aligned} \tag{3.65}$$

Thus, $\{x_n\}$ is Fejer monotone,

$$\sum_{n=1}^{\infty} \|T_{[n]}Ax_n - Ax_n\|^2 < \infty \text{ and } \sum_{n=1}^{\infty} \|U_{[n]}u_n - u_n\|^2 < \infty.$$

In particular, we have

$$\lim_{n\to\infty} \left\|U_{[n]}u_n - u_n\right\| = 0 \text{ and } \lim_{n\to\infty} \left\|T_{[n]}Ax_n - Ax_n\right\| = 0.$$

Finally, we show that $x_n \rightharpoonup x^*$. To show this, it suffices to show that $\omega_\omega \subset \Gamma$ (see Lemma 2.3). Let $x \in \omega_\omega(x_n)$, this implies that there exists a sub-sequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \rightharpoonup x$. Let $r \in \{1, 2, 3, ..., M\}$ such that $[n_k] := r$ for all k. By continuity of A, we have $Ax_{n_k} \rightharpoonup Ax$. It turns out that

$$\|T_r Ax_{n_k} - Ax_{n_k}\| = \left\|T_{[n_k]}Ax_{n_k} - Ax_{n_k}\right\| \to 0.$$

By demiclosedness of $(T_r - I)$ at zero, we deduce that $Ax \in Fix(T_r)$. Similarly, from the fact that $x_{n_k} \rightharpoonup x$, it follows that $u_{n_k} \rightharpoonup x$. By demiclosedness of $(U_r - I)$ at zero, we also deduce that $U_r x = x$ which implies that $x \in Fix(U_r)$. Hence, $x \in \Gamma$. This implies that $\omega_\omega \subset \Gamma$. By uniqueness of limit and Lemma 2.3, we conclude that $x_n \rightharpoonup x^*$. This completes the proof. ■

Theorem 3.7. *Let $T_r : H_1 \to H_1$ and $U_r : H_2 \to H_2$ for $r = 1, 2, 3, ..., M$ be k_r-strictly pseudocontractive mappings with $0 \le k_r < 1$, and let $k = \max\{k_r, 1 \le r \le M\}$, $A : H_1 \to H_2$ be a bounded linear operator with its adjoint A^*. Assume that for $N = M$, $\Gamma \neq \emptyset$, and let $\{x_n\}$ be the sequence defined by*

$$\begin{cases} x_1 \in C_1, \\ y_n = \alpha_n u_n + (1 - \alpha_n)U_{[n]}u_n, \\ u_n = x_n + \gamma A^*\big(T_{[n]} - I\big)Ax_n, \\ C_{n+1} \;= \Big\{z \in C_n : \|y_n - z\|^2 \le \|u_n - z\|^2 \le \|x_n - z\|^2\Big\}, \\ x_{n+1} \;= P_{C_{n+1}}(x_1), \forall n \ge 1, \end{cases} \tag{3.66}$$

where $[n] = n(mod\, M)$ with mod function taking values in $\{1, 2, 3, ..., M\}$, $k < \alpha_n < 1$ and $\gamma \in (0, \frac{1-k}{L})$ with $L = \|AA^*\|$. Then $x_n \rightharpoonup x^* \in \Gamma$.

Proof. The proof of this theorem follows directly from Theorem 3.4 by replacing $U_{[n]}$ with U and $T_{[n]}$ with T as in Algorithm (3.33). ■

The split common fixed point Problem (1.5) can be viewed as a special case of the common fixed point Problem (1.4), since (1.5) can be written as

$$x^* \in \bigcap_{s=1}^{N+M} Fix(U_s), \tag{3.67}$$

where $Fix(U_{N+r}) := \Big\{x^* \in H : x^* \in A^{-1}(Fix(T_r)),\ \text{for } 1 \le r \le M\Big\}$. By taking $\gamma = 0$ in Theorem 3.2, 3.3 and 3.4, we obtain the following corollaries.

Corollary 3.8. *Let $U : H \to H$ be a $k-$strictly pseudocontractive mapping with $k \in [0, 1)$. Assume that the common fixed point U is nonempty, and let $\{x_n\}$ be the sequence defined by*

$$\begin{cases} x_0 \text{ is chosen arbitrarily }, \\ x_{n+1} \;= \alpha_n x_n + (1 - \alpha_n)Ux_n, \forall n \ge 0, \end{cases} \tag{3.68}$$

where $k < \alpha_n < 1$ and $\sum(\alpha_n - k)(1 - \alpha_n) = \infty$. Then $\{x_n\}$ converges weakly to the common fixed point of U.

Proof. The proof of this corollary follows trivially from Theorem 3.1 by taking $\gamma = 0$. Algorithm (3.68) is known as Mann algorithm, see [16]. ■

Corollary 3.9. (Acedo and Xu [10]) *Let $U_s : H \to H$ be a k_s-strictly pseudocontractive mapping with $0 \le k_s < 1$ and let $k = \max\{k_s, 1 \le s \le N\}$ where $N \ge 1$. Assume that the common fixed set $\bigcap_{s=1}^{N} Fix(U_s)$ is nonempty, and let $\{\beta_s\}_{s=1}^{N}$ be a finite sequence of positive numbers such that $\sum_{s=1}^{N} \beta_s = 1$. Let also $\{x_n\}$ be the sequence generated by*

$$\begin{cases} x_0 \text{ is chosen arbitrarily;} \\ x_{n+1} = \alpha_n x_n + (1-\alpha_n)\sum_{s=1}^{N} \beta_s U_s x_n, \forall n \ge 0, \end{cases} \tag{3.69}$$

where $k < \alpha_n < 1$ for all n, and $\sum_{s=1}^{N} (\alpha_n - k)(1-\alpha_n) = \infty$. Then $\{x_n\}$ converges weakly to a common fixed point of $\{U_s\}_{s=1}^{N}$.

Corollary 3.10. (Acedo and Xu [1]) *Let $U_s : H \to H$ be a k_s-strictly pseudocontractive mapping with $0 \le k_s < 1$, and let $k = \max\{k_s, 1 \le s \le N\}$ where $N \ge 1$. Assume that the common fixed set $\bigcap_{s=1}^{N} Fix(U_s)$ is nonempty, and let $\{\beta_s\}_{s=1}^{N}$ be a finite sequence of positive numbers such that $\sum_{s=1}^{N} \beta_s^n = 1$ for all n and $\inf_{n\ge1} \beta_s^n$. Let also $\{x_n\}$ be the sequence defined by*

$$\begin{cases} x_0 \text{ is chosen arbitrarily;} \\ x_{n+1} = \alpha_n x_n + (1-\alpha_n)\sum_{s=1}^{N} \beta_s^n U_s x_n, \forall n \ge 0, \end{cases} \tag{3.70}$$

where $k < \alpha_n < 1$ for all n, such that $\sum (\alpha_n - k)(1-\alpha_n) = \infty$ and $\sum \sqrt{\sum_{s=1}^{N} |\beta_s^{n+1} - \beta_s^n|} < \infty$. Then $\{x_n\}$ converges weakly to a common fixed point of $\{U_s\}_{s=1}^{N}$.

Corollary 3.11. *Let C_1 be a nonempty closed convex subset of H_1, $T_r : C_1 \to H_1, 1 \le r \le M$, be k_r-strictly pseudocontractive mappings with $0 \le k_r < 1$. Let $k = \max\{k_r,$ for $r = 1, 2, 3, ..., M\}$ such that for all n, $k < \alpha_n < 1$. Assume that the common fixed set $\bigcap_{r=1}^{M} Fix(U_r)$ is nonempty and let $\{\lambda_r\}_{r=1}^{M}$ be finite sequence of positive number such that $\sum_{r=1}^{M} \lambda_r = 1$. Let also $\{x_n\}$ be the sequence generated by*

$$\begin{cases} x_1 \in C_1, \\ y_n = \alpha_n x_n + (1-\alpha_n)\sum_{r=1}^{M} \lambda_r U_r x_n, \\ C_{n+1} = \Big\{ z \in C_n : \|y_n - z\|^2 \le \|x_n - z\|^2 \Big\}, \\ x_{n+1} = P_{C_{n+1}}(x_1), \forall n \ge 1. \end{cases} \tag{3.71}$$

Then $\{x_n\}$ converges strongly to a common fixed point of $\{U_r\}_{r=1}^{M}$.

Corollary 3.12. *Let $T_r : H \to H$ for $r = 1, 2, 3, ..., M$ be k_r-strictly pseudocontractive mappings with $0 \le k_r < 1$, and let $k = \max\{k_r, 1 \le r \le M\}$. Assume that the common fixed set $\bigcap_{r=1}^{M} Fix(U_r)$ is nonempty and let $\{x_n\}$ be the sequence generated by*

$$\begin{cases} x_1 \in H_1; \\ x_{n+1} = \alpha_n x_n + (1-\alpha_n)U_{[n]} x_n, \forall n \ge 1, \end{cases} \tag{3.72}$$

where $[n] = n (\mathrm{mod}\, M)$ with mod function taking values in $\{1, 2, 3, ..., M\}$, $k < \alpha_n < 1$. Then $\{x_n\}$ converges weakly to a common fixed point of $\{U_r\}_{r=1}^{M}$.

Corollary 3.13. *Let C_1 be a nonempty closed convex subset of H_1, $T_r : C_1 \to H_1, 1 \le r \le M$, be k_r-strictly pseudocontractive mappings with $0 \le k_r < 1$. Let also $k = \max\{k_r, r =$*

$1, 2, 3, ..., M\}$. *Assume that the common fixed set* $\bigcap_{r=1}^{M} Fix(U_r)$ *is nonempty, and let* $\{x_n\}$ *be the sequence generated by*

$$\begin{cases} x_1 \in C_1, \\ y_n = \alpha_n x_n + (1-\alpha_n) U_{[n]} x_n, \\ C_{n+1} \ = \Big\{ z \in C_n : \|y_n - z\|^2 \le \|x_n - z\|^2 \Big\}, \\ x_{n+1} \ = P_{C_{n+1}}(x_1), \forall n \ge 1, \end{cases} \tag{3.73}$$

where $[n] = n(mod\, M)$ *with mod function taking values in* $\{1, 2, 3, ..., M\}$, $k < \alpha_n < 1$. *Then* $\{x_n\}$ *converges strongly to a common fixed point of* $\{U_r\}_{r=1}^{M}$.

Remark 3.14. By taking $k = 0$ in (1.9), we immediately obtained Equation (1.10) which is known as nonexpansive mapping. In Theorem 3.1, 3.2 and 3.3, as U and T are two nonexpansive mappings, we can also obtain similar results.

4. Application to Variational Inequality Problems

Let $T : C \to H_1$ be a nonlinear mapping. The variational inequality problem with respect to C consists as finding a vector $x^* \in C$ such that

$$\langle Tx^*, x - x^* \rangle \ge 0, \ \forall \ x \in C. \tag{4.1}$$

We denote the solution set of variational inequality problem (4.1) by $VI(T, C)$. It is easy to see that

$$x^* \in VI(T, C) \quad \text{if and only if} \quad x^* \in Fix(P_C(I - \beta T)), \tag{4.2}$$

where P_C is the metric projection from H_1 onto C and β is a positive constant. Let $Q := Fix(P_C(I - \beta T))$ (the fixed point set of $P_C(I - \beta T)$) and $A = I$ (the identity operator on H_1), then Equation (4.1) can be written as;

$$\text{find } x^* \in C \text{ such that } Ax^* \in Q. \tag{4.3}$$

Hence, as the consequence of Theorem 3.4, we have the following theorem.

Theorem 4.1. *Let* C_1 *be a nonempty closed convex subset of* H_1, $T_r : C_1 \to H_1, 1 \le r \le M$, *and* $U_s : H_2 \to H_2, 1 \le s \le N$ *be family of nonexpansive mappings. Assume that* $\Gamma \ne \emptyset$, *and let* $\{\lambda_s\}_{r=1}^{N}$ *and* $\{\beta_r\}_{r=1}^{M}$ *be finite sequences of positive numbers such that* $\sum_{s=1}^{N} \lambda_s = 1$ *and* $\sum_{r=1}^{M} \beta_r = 1$, *and also let* $\{x_n\}$ *be the sequence generated by*

$$\begin{cases} x_1 \in C_1, \\ y_n = \alpha_n u_n + (1-\alpha_n) \sum_{s=1}^{N} \lambda_s U_s u_n, \\ u_n = x_n + \gamma\big(\sum_{r=1}^{M} \beta_r T_r - I \big) x_n, \\ C_{n+1} \ = \Big\{ z \in C_n : \|y_n - z\|^2 \le \|u_n - z\|^2 \le \|x_n - z\|^2 \Big\}, \\ x_{n+1} \ = P_{C_{n+1}}(x_1), \forall n \ge 1, \end{cases} \tag{4.4}$$

where $0 < \alpha_n < 1$ *and* $\gamma \in (0, 1)$. *Then* $\{x_n\}$ *converges strongly to a solution of Problem (4.3).*

Proof. Since U and T are nonexpansive, then they are $k-$strictly pseudocontractive mapping with $k = 0$. Therefore, all the hypothesis of Theorem 3.4 are satisfied. Hence, the proof of this theorem follows directly from Theorem 3.4. ∎

Remark 4.2. The results presented in this paper, not only extend the results of Browder and Petryshyn [5], Acedo and Xu [1] and Marino and Xu [16] but also extend, improve and generalize several well-known results announced.

5. Numerical Example

In this section, we illustrate the convergence result of Theorem 3.1. Furthermore, we compare the convergence rate of Algorithm 3.1 with that of Browder and Petryshyn [5]. The following is an example of strictly pseudocontractive mappings.

Example 5.1. Let $H_1 = \mathbb{R}$ and $H_2 = \mathbb{R}$, $C := [0, \infty)$, and $Q := [0, \infty)$ be a subset of H_1 and H_2, respectively. Define $T : C \to C$ by $Tx = \frac{x+2}{3}$ for all $x \in C$ and $U : Q \to Q$ by

$$Ux = \begin{cases} \dfrac{2x}{x+1}, & \forall x \in (1, \infty) \\ 0, & \forall x \in [0, 1]. \end{cases} \tag{5.1}$$

Then, U and T are $k-$strictly pseudocontractive mappings. For any $x, y \in C$, we have that

$$\|Tx - Ty\|^2 = \frac{1}{4}\|x - y\|^2 \leq \|x - y\|^2 + k\|(x - Tx) - (y - Ty)\|^2.$$

Also, for any $x, y \in Q$, we have that

$$\|Ux - Uy\|^2 = \frac{2}{(1 + x)(1 + y)}\|x - y\|^2 \leq \|x - y\|^2 + k\|(x - Ux) - (y - Uy)\|^2.$$

Thus, U and T are $k-$strictly pseudocontractive mappings.

Example 5.2. Let $H_1 = \mathbb{R}$ and $H_2 = \mathbb{R}$, $C := [0, \infty)$, and $Q := [0, \infty)$ be subset of H_1 and H_2, respectively. Define $T : C \to C$ by $Tx = \frac{x+2}{3}$ for all $x \in C$, and $U : Q \to Q$ by

$$Ux = \begin{cases} \dfrac{2x}{x+1}, & \forall x \in (1, \infty) \\ 0, & \forall x \in [0, 1]. \end{cases} \tag{5.2}$$

Let also $Ax = x$, $\gamma = \frac{1}{4}$, $\alpha_n = \alpha = \frac{1}{5}$ and $\{x_n\}$ be the sequence defined by

$$\begin{cases} x_0 \in H_1 \text{ is chosen arbitrarily,} \\ u_n = x_n + \frac{1}{4}A^*\big(T - I\big)Ax_n, \\ x_{n+1} = \frac{1}{5}u_n + (1 - \frac{1}{5})Uu_n, \forall n \geq 0, \end{cases} \tag{5.3}$$

Then $\{x_n\}$ converges to $1 \in \Gamma$.

Proof. By Example 5.1, U and T are $k-$strictly pseudocontractive mappings with $Fix(U) = 1$ and $Fix(T) = 1$, respectively. Clearly, A is bounded linear on $\mathbb{R}$, and $A = A^* = 1$. Hence,

$$\Gamma = \{1 \in Fix(T) \text{ such that } A(1) \in Fix(U)\}.$$

After simplifying (5.3), we have

$$\begin{cases} x_0 \in H_1 \text{ is chosen arbitrarily,} \\ u_n = \frac{3x_n}{4} + \frac{x_n+2}{12}, \\ x_{n+1} = \frac{u_n}{5} + \frac{8u_n}{5(1+\ u_n)}, \forall n \geq 0. \end{cases} \tag{5.4}$$

■

TABLE 1. Numerical results of Example 5.3. Starting with initial values $x_0 = 10$ and $y_0 = 10$.

	Algorithm 5.3	Algorithm of [5]
n	x_n	y_n
0	10	10
1	3.131578947	5.200000000
2	1.731569779	2.960000000
3	1.308817257	1.914666666
.	.	.
.	.	.
14	1.000129227	1.000209092

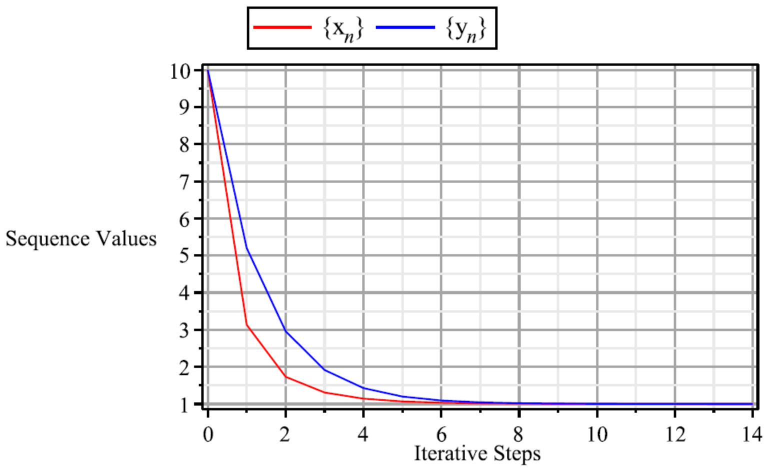

FIGURE 1. The convergence of $\{x_n\}$ for Algorithm 5.3 and $\{y_n\}$ for the algorithms given in [5] with initial values $x_0 = 10$ and $y_0 = 10$.

TABLE 2. Numerical results of Example 5.3. Starting with initial values $x_0 = -10$ and $y_0 = -10$.

	Algorithm 5.3	Algorithm of [5]
n	x_n	y_n
0	-10	-10
1	0.189922481	-4.133333333
2	0.4573810348	-1.395555556
3	0.6758499252	-0.1179259262
.	.	.
.	.	.
14	0.9998056078	0.9997444430

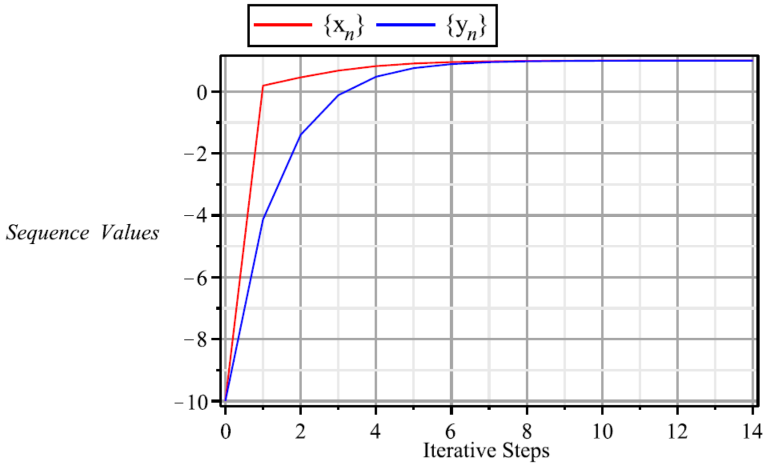

FIGURE 2. The convergence of $\{x_n\}$ for Algorithm 5.3 and $\{y_n\}$ for the algorithms given in [5] with initial values $x_0 = -10$ and $y_0 = -10$.

Remark 5.3. Table 1 and Figure 1 show that $\{x_n\}$ and $\{y_n\}$ converge to $1 \in \Gamma$. It also indicates that the convergence of Algorithm 5.3 is faster than those algorithms given in [5]. Similarly, Table 2 and Figure 2 show that $\{x_n\}$ and $\{y_n\}$ converge to $1 \in \Gamma$. Furthermore, Table 2 indicated that the convergence of Algorithm 5.3 is faster than those algorithms given in [5].

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