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Contributions to three-dimensional Hardy-Hilbert-type integral inequalities

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Abstract This article presents new extensions of the classical Hardy-Hilbert integral inequality involving primitive functions in three dimensions. Two complementary theorems are established, each providing a sharp upper bound for triple integrals associated with distinct kernel function structures. By introducing additional parameters, the proposed results offer greater flexibility and generality in the formulation and application of these inequalities.

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1. Introduction

The classical Hardy-Hilbert integral inequality is a key result in the theory of integral inequalities. Its precise form is stated below. Let $p > 1$ and q be such that $1/p + 1/q = 1$, and $f, g : (0, +\infty) \to (0, +\infty)$ be two functions. Then we have

$$\int_0^{+\infty} \int_0^{+\infty} \frac{1}{x+y} f(x)g(y)dxdy \\ \leq \frac{\pi}{\sin(\pi/p)} \left(\int_0^{+\infty} f^p(x)dx \right)^{1/p} \left(\int_0^{+\infty} g^q(y)dy \right)^{1/q},$$

provided that the integrals exist. This inequality provides a sharp upper bound for the double integral in terms of the integral norms of the functions f and g. It serves as a classical benchmark for numerous extensions and generalizations (see, e.g., [4, 8]). One notable direction of extension involves replacing f and g by their primitive functions within the double integral, leading to new forms of Hardy-Hilbert-type integral inequalities. Several such results appear in [1–3, 5–7]. In particular, focusing on three dimensions,

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[7, Theorem 2] is stated below. Let $\alpha, \beta, \gamma > 0$, $p, q, r > 1$ be such that $1/p+1/q+1/r = 1$ and $p < \alpha/4$, $q < \beta/4$ and $r < \gamma/4$, $f, g, h : (0, +\infty) \to (0, +\infty)$ be three functions, and

$$F(x) = \int_0^x f(t)dt, \quad G(y) = \int_0^y g(t)dt, \quad H(z) = \int_0^z h(t)dt,$$

provided that they exist. Then we have

$$\begin{aligned}
&\int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty} \frac{x^{2/\beta+1/\gamma-1}y^{2/\gamma+1/\alpha-1}z^{2/\alpha+1/\beta-1}}{|x-y-z|^{4/\alpha+4/\beta+4/\gamma}} \\
&\quad \times F^{1/\alpha+1/p}(x)G^{1/\beta+1/q}(y)H^{1/\gamma+1/r}(z)dxdydz \\
&\leq 4^{1/p}B^{1/p}\left(\frac{p}{\alpha}, 1-2\frac{p}{\alpha}\right)B^{1/p}\left(2\frac{p}{\alpha}, 1-4\frac{p}{\alpha}\right)\alpha^{1/\alpha+1/p}\left(\frac{1}{\alpha}+\frac{1}{p}\right)^{1/\alpha+1/p} \\
&\quad \times 2^{1/q}B^{1/q}\left(\frac{q}{\beta}, \frac{q}{\beta}\right)B^{1/q}\left(2\frac{q}{\beta}, 1-4\frac{q}{\beta}\right)\beta^{1/\beta+1/q}\left(\frac{1}{\beta}+\frac{1}{q}\right)^{1/\beta+1/q} \\
&\quad \times 4^{1/r}B^{1/r}\left(\frac{r}{\gamma}, 1-2\frac{r}{\gamma}\right)B^{1/r}\left(2\frac{r}{\gamma}, 1-4\frac{r}{\gamma}\right)\gamma^{1/\gamma+1/r}\left(\frac{1}{\gamma}+\frac{1}{r}\right)^{1/\gamma+1/r} \\
&\quad \times \left(\int_0^{+\infty} f^{p/\alpha+1}(x)dx\right)^{1/p}\left(\int_0^{+\infty} g^{q/\beta+1}(y)dy\right)^{1/q}\left(\int_0^{+\infty} h^{r/\gamma+1}(z)dz\right)^{1/r},
\end{aligned}$$

provided that the integrals exist. This result is significant as it extends the classical Hardy-Hilbert integral inequality to a three-dimensional framework involving primitive functions. It provides an exact upper bound for a triple integral with a highly non-symmetric kernel function and establishes a clear connection between the integrals of f, g, and h and those of their primitive functions F, G, and H, respectively. Such multidimensional extensions play a crucial role in the advancement of modern integral inequality theory and find applications in various areas of analysis and partial differential equations.

In this article, inspired by the aforementioned result, we contribute to this line of research in two complementary directions.

First, we investigate an upper bound for a triple integral of the form

$$\begin{aligned}
&\int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty} \frac{x^{2/\beta+1/\gamma-1}y^{2/\gamma+1/\alpha-1}z^{2/\alpha+1/\beta-1}}{(x+y+z)^{4/\alpha+4/\beta+4/\gamma}} \\
&\times F^{\eta+1/p}(x)G^{\theta+1/q}(y)H^{\omega+1/r}(z)dxdydz.
\end{aligned}$$

Compared to [7, Theorem 2], the main novelty lies in the introduction of the parameters η, θ, and ω, which provide additional flexibility in the functional structure. Furthermore, the denominator of the kernel function, $(x+y+z)^{4/\alpha+4/\beta+4/\gamma}$, introduces a distinct symmetry and differs fundamentally from the denominator $|x-y-z|^{4/\alpha+4/\beta+4/\gamma}$ considered in [7]. This modification yields a more balanced and globally symmetric structure, which aligns more closely with the framework of the classical three-dimensional Hardy-Hilbert integral inequality. Compared to [2, Proposition 3.2], the numerator is simpler, and the parameters offer greater flexibility, representing a significant improvement in this regard. Consequently, the constant factor is entirely different, reflecting these structural changes.

Second, we investigate an upper bound for a triple integral of the form

$$\int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty} \frac{x^{2/\beta+1/\gamma-1}y^{2/\gamma+1/\alpha-1}z^{2/\alpha+1/\beta-1}}{|x-y-z|^{4/\alpha+4/\beta+4/\gamma}} \times F^{\eta+1/p}(x)G^{\theta+1/q}(y)H^{\omega+1/r}(z)dxdydz.$$

This result extends [7, Theorem 2] by incorporating the additional parameters η, θ, and ω, which provide increased flexibility in the formulation of the inequality.

Together, these two results establish broader families of Hardy-Hilbert-type integral inequalities, offering new insights and potential applications in higher-dimensional integral analysis.

The remainder of this article is organized as follows: Section 2 presents the first main theorem together with its proof. Section 3 is devoted to the second theorem and its proof. Finally, Section 4 provides concluding remarks.

2. First theorem

The first theorem below establishes a sharp upper bound for a triple integral of the form

$$\int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty} \frac{x^{2/\beta+1/\gamma-1}y^{2/\gamma+1/\alpha-1}z^{2/\alpha+1/\beta-1}}{(x+y+z)^{4/\alpha+4/\beta+4/\gamma}} \times F^{\eta+1/p}(x)G^{\theta+1/q}(y)H^{\omega+1/r}(z)dxdydz.$$

A detailed proof follows, relying on classical analytical tools and integral inequalities, including [4, Theorem 330].

Theorem 2.1. *Let $\alpha,\beta,\gamma,\eta,\theta,\omega>0$, $p,q,r>1$ be such that $1/p+1/q+1/r=1$, $f,g,h:(0,+\infty)\to(0,+\infty)$ be three functions, and*

$$F(x)=\int_0^x f(t)dt,\quad G(y)=\int_0^y g(t)dt,\quad H(z)=\int_0^z h(t)dt,$$

provided that they exist. Then we have

$$\begin{aligned}
&\int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty} \frac{x^{2/\beta+1/\gamma-1}y^{2/\gamma+1/\alpha-1}z^{2/\alpha+1/\beta-1}}{(x+y+z)^{4/\alpha+4/\beta+4/\gamma}} \\
&\quad\times F^{\eta+1/p}(x)G^{\theta+1/q}(y)H^{\omega+1/r}(z)dxdydz \\
&\le B^{1/p}\left(\frac{p}{\alpha},\frac{p}{\alpha}\right)B^{1/p}\left(2\frac{p}{\alpha},2\frac{p}{\alpha}\right)\alpha^{\eta+1/p}\left(\eta+\frac{1}{p}\right)^{\eta+1/p} \\
&\quad\times B^{1/q}\left(\frac{q}{\beta},\frac{q}{\beta}\right)B^{1/q}\left(2\frac{q}{\beta},2\frac{q}{\beta}\right)\beta^{\theta+1/q}\left(\theta+\frac{1}{q}\right)^{\theta+1/q} \\
&\quad\times B^{1/r}\left(\frac{r}{\gamma},\frac{r}{\gamma}\right)B^{1/r}\left(2\frac{r}{\gamma},2\frac{r}{\gamma}\right)\gamma^{\omega+1/r}\left(\omega+\frac{1}{r}\right)^{\omega+1/r} \\
&\quad\times\left(\int_0^{+\infty} x^{(\eta-1/\alpha)p}f^{\eta p+1}(x)dx\right)^{1/p}\left(\int_0^{+\infty} y^{(\theta-1/\beta)q}g^{\theta q+1}(y)dy\right)^{1/q} \\
&\quad\times\left(\int_0^{+\infty} z^{(\omega-1/\gamma)r}h^{\omega r+1}(z)dz\right)^{1/r},
\end{aligned}$$

provided that the integrals exist, where, for any $u, v > 0$, $B(u, v)$ denotes the standard beta function given by the two following integral representations:

$$B(u,v) = \int_0^1 t^{u-1}(1-t)^{v-1}dt = \int_0^{+\infty} \frac{z^{v-1}}{(1+z)^{u+v}}dz.$$

Proof. By decomposing the integrand in an appropriate manner and applying the Hölder integral inequality with the exponents p, q, and r, we get

$$\begin{aligned}
&\int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty} \frac{x^{2/\beta+1/\gamma-1}y^{2/\gamma+1/\alpha-1}z^{2/\alpha+1/\beta-1}}{(x+y+z)^{4/\alpha+4/\beta+4/\gamma}} \\
&\quad \times F^{\eta+1/p}(x)G^{\theta+1/q}(y)H^{\omega+1/r}(z)dxdydz \\
&= \int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty} \frac{y^{1/\alpha-1/p}z^{2/\alpha-1/p}}{x^{1/p}(x+y+z)^{4/\alpha}}F^{\eta+1/p}(x) \\
&\quad \times \frac{z^{1/\beta-1/q}x^{2/\beta-1/q}}{y^{1/q}(x+y+z)^{4/\beta}}G^{\theta+1/q}(y)\frac{x^{1/\gamma-1/r}y^{2/\gamma-1/r}}{z^{1/r}(x+y+z)^{4/\gamma}}H^{\omega+1/r}(z)dxdydz \\
&\leq A^{1/p}B^{1/q}C^{1/r}, \qquad (2.1)
\end{aligned}$$

where

$$A = \int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty} \frac{y^{p/\alpha-1}z^{2p/\alpha-1}}{x(x+y+z)^{4p/\alpha}}F^{\eta p+1}(x)dxdydz,$$

$$B = \int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty} \frac{z^{q/\beta-1}x^{2q/\beta-1}}{y(x+y+z)^{4q/\beta}}G^{\theta q+1}(y)dxdydz$$

and

$$C = \int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty} \frac{x^{r/\gamma-1}y^{2r/\gamma-1}}{z(x+y+z)^{4r/\gamma}}H^{\omega r+1}(z)dxdydz.$$

Let us find an upper bound for A. Applying the Fubini-Tonelli integral theorem, making the changes of variables $u = z/(x+y)$ and $v = y/x$, and using the beta function, we obtain

$$\begin{aligned}
A &= \int_0^{+\infty} x^{-p/\alpha-1}F^{\eta p+1}(x)\int_0^{+\infty} \frac{(y/x)^{p/\alpha-1}(1/x)}{(1+y/x)^{2p/\alpha}} \\
&\quad \times \int_0^{+\infty} \frac{(z/(x+y))^{2p/\alpha-1}(1/(x+y))}{(1+z/(x+y))^{4p/\alpha}}dzdydx \\
&= \int_0^{+\infty} x^{-p/\alpha-1}F^{\eta p+1}(x)\int_0^{+\infty} \frac{v^{p/\alpha-1}}{(1+v)^{2p/\alpha}}dv\int_0^{+\infty} \frac{u^{2p/\alpha-1}}{(1+u)^{4p/\alpha}}dudx \\
&= B\left(\frac{p}{\alpha},\frac{p}{\alpha}\right)B\left(2\frac{p}{\alpha},2\frac{p}{\alpha}\right)\int_0^{+\infty} x^{-p/\alpha-1}F^{\eta p+1}(x)dx.
\end{aligned}$$

The remaining integral can be bound by using a general result in [4], as stated below. Let $r, s > 1$. Then the generalized version of the Hardy integral inequality stated in [4, Theorem 330] ensures that

$$\int_0^{+\infty} x^{-r}F^s(x)dx \leq \left(\frac{s}{r-1}\right)^s \int_0^{+\infty} x^{s-r}f^s(x)dx. \qquad (2.2)$$

Applying this to $r = p/\alpha + 1$ and $s = \eta p + 1$, after some simplifications, we obtain

$$\int_0^{+\infty} x^{-p/\alpha-1} F^{\eta p+1}(x)dx \le \alpha^{\eta p+1} \left(\eta + \frac{1}{p}\right)^{\eta p+1} \int_0^{+\infty} x^{(\eta-1/\alpha)p} f^{\eta p+1}(x)dx,$$

which yields

$$A \le B\left(\frac{p}{\alpha}, \frac{p}{\alpha}\right) B\left(2\frac{p}{\alpha}, 2\frac{p}{\alpha}\right) \alpha^{\eta p+1} \left(\eta + \frac{1}{p}\right)^{\eta p+1} \int_0^{+\infty} x^{(\eta-1/\alpha)p} f^{\eta p+1}(x)dx. \tag{2.3}$$

Let us now find an upper bound for B. Applying the Fubini-Tonelli integral theorem, making the changes of variables $u = x/(y+z)$ and $v = z/y$, and using the beta function, we get

$$\begin{aligned} B &= \int_0^{+\infty} y^{-q/\beta-1} G^{\theta q+1}(y) \int_0^{+\infty} \frac{(z/y)^{q/\beta-1}(1/y)}{(1+z/y)^{2q/\beta}} \\ &\quad \times \int_0^{+\infty} \frac{(x/(y+z))^{2q/\beta-1}(1/(y+z))}{(1+x/(y+z))^{4q/\beta}} dxdzdy \\ &= \int_0^{+\infty} y^{-q/\beta-1} G^{\theta q+1}(y) \int_0^{+\infty} \frac{v^{q/\beta-1}}{(1+v)^{2q/\beta}} dv \int_0^{+\infty} \frac{u^{2q/\beta-1}}{(1+u)^{4q/\beta}} dudy \\ &= B\left(\frac{q}{\beta}, \frac{q}{\beta}\right) B\left(2\frac{q}{\beta}, 2\frac{q}{\beta}\right) \int_0^{+\infty} y^{-q/\beta-1} G^{\theta q+1}(y)dy. \end{aligned}$$

Applying Equation (2.2) to $r = q/\beta + 1$ and $s = \theta q + 1$, after some simplifications, we obtain

$$\int_0^{+\infty} y^{-q/\beta-1} G^{\theta q+1}(y)dy \le \beta^{\theta q+1} \left(\theta + \frac{1}{q}\right)^{\theta q+1} \int_0^{+\infty} y^{(\theta-1/\beta)q} g^{\theta q+1}(y)dy,$$

which yields

$$B \le B\left(\frac{q}{\beta}, \frac{q}{\beta}\right) B\left(2\frac{q}{\beta}, 2\frac{q}{\beta}\right) \beta^{\theta q+1} \left(\theta + \frac{1}{q}\right)^{\theta q+1} \int_0^{+\infty} y^{(\theta-1/\beta)q} g^{\theta q+1}(y)dy. \tag{2.4}$$

Finally, let us find an upper bound for C. Applying the Fubini-Tonelli integral theorem, making the changes of variables $u = y/(x+z)$ and $v = x/z$, and using the beta function, we derive

$$\begin{aligned} C &= \int_0^{+\infty} z^{-r/\gamma-1} H^{\omega r+1}(z) \int_0^{+\infty} \frac{(x/z)^{r/\gamma-1}(1/z)}{(1+x/z)^{2r/\gamma}} \\ &\quad \times \int_0^{+\infty} \frac{(y/(x+z))^{2r/\gamma-1}(1/(x+z))}{(1+y/(x+z))^{4r/\gamma}} dydxdz \\ &= \int_0^{+\infty} z^{-r/\gamma-1} H^{\omega r+1}(z) \int_0^{+\infty} \frac{v^{r/\gamma-1}}{(1+v)^{2r/\gamma}} dv \int_0^{+\infty} \frac{u^{2r/\gamma-1}}{(1+u)^{4r/\gamma}} dudz \\ &= B\left(\frac{r}{\gamma}, \frac{r}{\gamma}\right) B\left(2\frac{r}{\gamma}, 2\frac{r}{\gamma}\right) \int_0^{+\infty} z^{-r/\gamma-1} H^{\omega r+1}(z)dz. \end{aligned}$$

Applying Equation (2.2) to $r = r/\gamma + 1$ and $s = \omega r + 1$, after some simplifications, we obtain

$$\int_0^{+\infty} z^{-r/\gamma-1} H^{\omega r+1}(z)dz \leq \gamma^{\omega r+1}\left(\omega + \frac{1}{r}\right)^{\omega r+1}\int_0^{+\infty} z^{(\omega-1/\gamma)r} h^{\omega r+1}(z)dz,$$

which yields

$$C \leq B\left(\frac{r}{\gamma}, \frac{r}{\gamma}\right) B\left(2\frac{r}{\gamma}, 2\frac{r}{\gamma}\right)\gamma^{\omega r+1}\left(\omega + \frac{1}{r}\right)^{\omega r+1}\int_0^{+\infty} z^{(\omega-1/\gamma)r} h^{\omega r+1}(z)dz. \tag{2.5}$$

Combining Equations (2.1), (2.3), (2.4), and (2.5), we find that

$$\begin{aligned}
&\int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty} \frac{x^{2/\beta+1/\gamma-1}y^{2/\gamma+1/\alpha-1}z^{2/\alpha+1/\beta-1}}{(x+y+z)^{4/\alpha+4/\beta+4/\gamma}} \\
&\quad \times F^{\eta+1/p}(x)G^{\theta+1/q}(y)H^{\omega+1/r}(z)dxdydz \\
&\leq B^{1/p}\left(\frac{p}{\alpha}, \frac{p}{\alpha}\right) B^{1/p}\left(2\frac{p}{\alpha}, 2\frac{p}{\alpha}\right)\alpha^{\eta+1/p}\left(\eta + \frac{1}{p}\right)^{\eta+1/p} \\
&\quad \times B^{1/q}\left(\frac{q}{\beta}, \frac{q}{\beta}\right) B^{1/q}\left(2\frac{q}{\beta}, 2\frac{q}{\beta}\right)\beta^{\theta+1/q}\left(\theta + \frac{1}{q}\right)^{\theta+1/q} \\
&\quad \times B^{1/r}\left(\frac{r}{\gamma}, \frac{r}{\gamma}\right) B^{1/r}\left(2\frac{r}{\gamma}, 2\frac{r}{\gamma}\right)\gamma^{\omega+1/r}\left(\omega + \frac{1}{r}\right)^{\omega+1/r} \\
&\quad \times \left(\int_0^{+\infty} x^{(\eta-1/\alpha)p} f^{\eta p+1}(x)dx\right)^{1/p}\left(\int_0^{+\infty} y^{(\theta-1/\beta)q} g^{\theta q+1}(y)dy\right)^{1/q} \\
&\quad \times \left(\int_0^{+\infty} z^{(\omega-1/\gamma)r} h^{\omega r+1}(z)dz\right)^{1/r}.
\end{aligned}$$

This completes the proof. ■

Let us now discuss some special cases of this theorem. By taking $\eta = 1/\alpha$, $\theta = 1/\beta$ and $\omega = 1/\gamma$, we get

$$\begin{aligned}
&\int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty} \frac{x^{2/\beta+1/\gamma-1}y^{2/\gamma+1/\alpha-1}z^{2/\alpha+1/\beta-1}}{(x+y+z)^{4/\alpha+4/\beta+4/\gamma}} \\
&\quad F^{1/\alpha+1/p}(x)G^{1/\beta+1/q}(y)H^{1/\gamma+1/r}(z)dxdydz \\
&\leq B^{1/p}\left(\frac{p}{\alpha}, \frac{p}{\alpha}\right) B^{1/p}\left(2\frac{p}{\alpha}, 2\frac{p}{\alpha}\right)\alpha^{1/\alpha+1/p}\left(\frac{1}{\alpha} + \frac{1}{p}\right)^{\eta+1/p} \\
&\quad \times B^{1/q}\left(\frac{q}{\beta}, \frac{q}{\beta}\right) B^{1/q}\left(2\frac{q}{\beta}, 2\frac{q}{\beta}\right)\beta^{1/\beta+1/q}\left(\frac{1}{\beta} + \frac{1}{q}\right)^{\theta+1/q} \\
&\quad \times B^{1/r}\left(\frac{r}{\gamma}, \frac{r}{\gamma}\right) B^{1/r}\left(2\frac{r}{\gamma}, 2\frac{r}{\gamma}\right)\gamma^{1/\gamma+1/r}\left(\frac{1}{\gamma} + \frac{1}{r}\right)^{\omega+1/r} \\
&\quad \times \left(\int_0^{+\infty} f^{p/\alpha+1}(x)dx\right)^{1/p}\left(\int_0^{+\infty} g^{q/\beta+1}(y)dy\right)^{1/q}\left(\int_0^{+\infty} h^{r/\gamma+1}(z)dz\right)^{1/r}.
\end{aligned}$$

The interest of this integral inequality is that the integral norms of f, g, and h are unweighted.

On the other hand, by taking $\eta = 1/q + 1/r$, $\theta = 1/p + 1/r$ and $\omega = 1/p + 1/q$, we obtain the following elegant inequality:

$$\begin{aligned}
&\int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty} \frac{x^{2/\beta+1/\gamma-1}y^{2/\gamma+1/\alpha-1}z^{2/\alpha+1/\beta-1}}{(x+y+z)^{4/\alpha+4/\beta+4/\gamma}} F(x)G(y)H(z)dxdydz \\
&\leq B^{1/p}\left(\frac{p}{\alpha},\frac{p}{\alpha}\right)B^{1/p}\left(2\frac{p}{\alpha},2\frac{p}{\alpha}\right)\alpha B^{1/q}\left(\frac{q}{\beta},\frac{q}{\beta}\right)B^{1/q}\left(2\frac{q}{\beta},2\frac{q}{\beta}\right)\beta \\
&\quad\times B^{1/r}\left(\frac{r}{\gamma},\frac{r}{\gamma}\right)B^{1/r}\left(2\frac{r}{\gamma},2\frac{r}{\gamma}\right)\gamma \\
&\quad\times\left(\int_0^{+\infty} x^{(1-1/\alpha)p-1}f^p(x)dx\right)^{1/p}\left(\int_0^{+\infty} y^{(1-1/\beta)q-1}g^q(y)dy\right)^{1/q} \\
&\quad\times\left(\int_0^{+\infty} z^{(1-1/\gamma)r-1}h^r(z)dz\right)^{1/r}.
\end{aligned}$$

Note that F, G, and H are considered directly within the triple integral; they are raised to the power 1. To the best of the knowledge of the author, these integral inequalities are new.

3. Second theorem

The second theorem below provides an upper bound for a triple integral of the form

$$\begin{aligned}
&\int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty} \frac{x^{2/\beta+1/\gamma-1}y^{2/\gamma+1/\alpha-1}z^{2/\alpha+1/\beta-1}}{|x-y-z|^{4/\alpha+4/\beta+4/\gamma}} \\
&\times F^{\eta+1/p}(x)G^{\theta+1/q}(y)H^{\omega+1/r}(z)dxdydz.
\end{aligned}$$

A detailed proof follows, relying on classical analytical tools and integral inequalities, including [4, Theorem 330] and [7, Lemma 1].

Theorem 3.1. *Let $\alpha,\beta,\gamma,\eta,\theta,\omega > 0$, $p,q,r > 1$ be such that $1/p+1/q+1/r = 1$ and $p < \alpha/4$, $q < \beta/4$ and $r < \gamma/4$, $f,g,h : (0,+\infty) \to (0,+\infty)$ be three functions, and*

$$F(x) = \int_0^x f(t)dt, \quad G(y) = \int_0^y g(t)dt, \quad H(z) = \int_0^z h(t)dt,$$

provided that they exist. Then we have

$$\int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty}\frac{x^{2/\beta+1/\gamma-1}y^{2/\gamma+1/\alpha-1}z^{2/\alpha+1/\beta-1}}{|x-y-z|^{4/\alpha+4/\beta+4/\gamma}}$$
$$\times F^{\eta+1/p}(x)G^{\theta+1/q}(y)H^{\omega+1/r}(z)dxdydz$$
$$\leq 4^{1/p}B^{1/p}\left(\frac{p}{\alpha},1-2\frac{p}{\alpha}\right)B^{1/p}\left(2\frac{p}{\alpha},1-4\frac{p}{\alpha}\right)\alpha^{\eta+1/p}\left(\eta+\frac{1}{p}\right)^{\eta+1/p}$$
$$\times 2^{1/q}B^{1/q}\left(\frac{q}{\beta},\frac{q}{\beta}\right)B^{1/q}\left(2\frac{q}{\beta},1-4\frac{q}{\beta}\right)\beta^{\theta+1/q}\left(\theta+\frac{1}{q}\right)^{\theta+1/q}$$
$$\times 4^{1/r}B^{1/r}\left(\frac{r}{\gamma},1-2\frac{r}{\gamma}\right)B^{1/r}\left(2\frac{r}{\gamma},1-4\frac{r}{\gamma}\right)\gamma^{\omega+1/r}\left(\omega+\frac{1}{r}\right)^{\omega+1/r}$$
$$\times\left(\int_0^{+\infty}x^{(\eta-1/\alpha)p}f^{\eta p+1}(x)dx\right)^{1/p}\left(\int_0^{+\infty}y^{(\theta-1/\beta)q}g^{\theta q+1}(y)dy\right)^{1/q}$$
$$\times\left(\int_0^{+\infty}z^{(\omega-1/\gamma)r}h^{\omega r+1}(z)dz\right)^{1/r},$$

provided that the integrals exist.

Proof. By decomposing the integrand in an appropriate manner and applying the Hölder integral inequality with the exponents p, q, and r, we get

$$\int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty}\frac{x^{2/\beta+1/\gamma-1}y^{2/\gamma+1/\alpha-1}z^{2/\alpha+1/\beta-1}}{|x-y-z|^{4/\alpha+4/\beta+4/\gamma}}$$
$$F^{\eta+1/p}(x)G^{\theta+1/q}(y)H^{\omega+1/r}(z)dxdydz$$
$$=\int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty}\frac{y^{1/\alpha-1/p}z^{2/\alpha-1/p}}{x^{1/p}|x-y-z|^{4/\alpha}}F^{\eta+1/p}(x)$$
$$\times\frac{z^{1/\beta-1/q}x^{2/\beta-1/q}}{y^{1/q}|x-y-z|^{4/\beta}}G^{\theta+1/q}(y)\frac{x^{1/\gamma-1/r}y^{2/\gamma-1/r}}{z^{1/r}|x-y-z|^{4/\gamma}}H^{\omega+1/r}(z)dxdydz$$
$$\leq D^{1/p}E^{1/q}F^{1/r}, \tag{3.1}$$

where

$$D=\int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty}\frac{y^{p/\alpha-1}z^{2p/\alpha-1}}{x|x-y-z|^{4p/\alpha}}F^{\eta p+1}(x)dxdydz,$$

$$E=\int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty}\frac{z^{q/\beta-1}x^{2q/\beta-1}}{y|x-y-z|^{4q/\beta}}G^{\theta q+1}(y)dxdydz$$

and

$$F=\int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty}\frac{x^{r/\gamma-1}y^{2r/\gamma-1}}{z|x-y-z|^{4r/\gamma}}H^{\omega r+1}(z)dxdydz.$$

Let us find an upper bound for D. Using the triangle inequality, i.e., $||x-y|-z|\leq|x-y-z|$, applying the Fubini-Tonelli integral theorem, making the changes of variables

$u = z/|x-y|$ and $v = y/x$, and using [7, Lemma 1], we obtain

$$\begin{aligned}
D &\le \int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty} \frac{y^{p/\alpha-1}z^{2p/\alpha-1}}{x\,||x-y|-z|^{4p/\alpha}}F^{\eta p+1}(x)dxdydz\\
&= \int_0^{+\infty} x^{-p/\alpha-1}F^{\eta p+1}(x)\int_0^{+\infty}\frac{(y/x)^{p/\alpha-1}(1/x)}{|1-y/x|^{2p/\alpha}}\\
&\quad\times\int_0^{+\infty}\frac{(z/|x-y|)^{2p/\alpha-1}\,(1/|x-y|)}{|1-z/|x-y||^{4p/\alpha}}dzdydx\\
&= \int_0^{+\infty} x^{-p/\alpha-1}F^{\eta p+1}(x)\int_0^{+\infty}\frac{v^{p/\alpha-1}}{|1-v|^{2p/\alpha}}dv\int_0^{+\infty}\frac{u^{2p/\alpha-1}}{|1-u|^{4p/\alpha}}dudx\\
&= 4B\left(\frac{p}{\alpha},1-2\frac{p}{\alpha}\right)B\left(2\frac{p}{\alpha},1-4\frac{p}{\alpha}\right)\int_0^{+\infty} x^{-p/\alpha-1}F^{\eta p+1}(x)dx.
\end{aligned}$$

Applying Equation (2.2) to $r = p/\alpha + 1$ and $s = \eta p + 1$, after some simplifications, we obtain

$$\int_0^{+\infty} x^{-p/\alpha-1}F^{\eta p+1}(x)dx \le \alpha^{\eta p+1}\left(\eta+\frac{1}{p}\right)^{\eta p+1}\int_0^{+\infty} x^{(\eta-1/\alpha)p}f^{\eta p+1}(x)dx,$$

which yields

$$\begin{aligned}
D \le\; & 4B\left(\frac{p}{\alpha},1-2\frac{p}{\alpha}\right)B\left(2\frac{p}{\alpha},1-4\frac{p}{\alpha}\right)\alpha^{\eta p+1}\left(\eta+\frac{1}{p}\right)^{\eta p+1}\\
&\times\int_0^{+\infty} x^{(\eta-1/\alpha)p}f^{\eta p+1}(x)dx. \qquad (3.2)
\end{aligned}$$

Let us now find an upper bound for E. Applying the Fubini-Tonelli integral theorem, making the changes of variables $u = x/(y+z)$ and $v = z/y$, and using the beta function and [7, Lemma 1], we derive

$$\begin{aligned}
E &= \int_0^{+\infty} y^{-q/\beta-1}G^{\theta q+1}(y)\int_0^{+\infty}\frac{(z/y)^{q/\beta-1}(1/y)}{(1+z/y)^{2q/\beta}}\\
&\quad\times\int_0^{+\infty}\frac{(x/(y+z))^{2q/\beta-1}\,(1/(y+z))}{|1-x/(y+z)|^{4q/\beta}}dxdzdy\\
&= \int_0^{+\infty} y^{-q/\beta-1}G^{\theta q+1}(y)\int_0^{+\infty}\frac{v^{q/\beta-1}}{(1+v)^{2q/\beta}}dv\int_0^{+\infty}\frac{u^{2q/\beta-1}}{|1-u|^{4q/\beta}}dudy\\
&= 2B\left(\frac{q}{\beta},\frac{q}{\beta}\right)B\left(2\frac{q}{\beta},1-4\frac{q}{\beta}\right)\int_0^{+\infty} y^{-q/\beta-1}G^{\theta q+1}(y)dy.
\end{aligned}$$

Applying Equation (2.2) to $r = q/\beta + 1$ and $s = \theta q + 1$, after some simplifications, we obtain

$$\int_0^{+\infty} y^{-q/\beta-1}G^{\theta q+1}(y)dy \le \beta^{\theta q+1}\left(\theta+\frac{1}{q}\right)^{\theta q+1}\int_0^{+\infty} y^{(\theta-1/\beta)q}g^{\theta q+1}(y)dy,$$

which yields

$$E \leq 2B\left(\frac{q}{\beta},\frac{q}{\beta}\right)B\left(2\frac{q}{\beta},1-4\frac{q}{\beta}\right)\beta^{\theta q+1}\left(\theta+\frac{1}{q}\right)^{\theta q+1}\int_0^{+\infty} y^{(\theta-1/\beta)q}g^{\theta q+1}(y)dy. \tag{3.3}$$

Finally, let us find an upper bound for F. Using the triangle inequality, applying the Fubini-Tonelli integral theorem, making the changes of variables $u = y/|x-z|$ and $v = x/z$, and using the beta function, we get

$$\begin{aligned} F &\leq \int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty} \frac{x^{r/\gamma-1}y^{2r/\gamma-1}}{z\,||x-z|-y|^{4r/\gamma}}H^{\omega r+1}(z)dxdydz \\ &= \int_0^{+\infty} z^{-r/\gamma-1}H^{\omega r+1}(z)\int_0^{+\infty}\frac{(x/z)^{r/\gamma-1}(1/z)}{|1-x/z|^{2r/\gamma}} \\ &\quad\times\int_0^{+\infty}\frac{(y/|x-z|)^{2r/\gamma-1}(1/|x-z|)}{|1-y/|x-z||^{4r/\gamma}}dydxdz \\ &= \int_0^{+\infty} z^{-r/\gamma-1}H^{\omega r+1}(z)\int_0^{+\infty}\frac{v^{r/\gamma-1}}{|1-v|^{2r/\gamma}}dv\int_0^{+\infty}\frac{u^{2r/\gamma-1}}{|1-u|^{4r/\gamma}}dudz \\ &= 4B\left(\frac{r}{\gamma},1-2\frac{r}{\gamma}\right)B\left(2\frac{r}{\gamma},1-4\frac{r}{\gamma}\right)\int_0^{+\infty} z^{-r/\gamma-1}H^{\omega r+1}(z)dz. \end{aligned}$$

Applying Equation (2.2) to $r = r/\gamma+1$ and $s = \omega r+1$, after some simplifications, we obtain

$$\int_0^{+\infty} z^{-r/\gamma-1}H^{\omega r+1}(z)dz \leq \gamma^{\omega r+1}\left(\omega+\frac{1}{r}\right)^{\omega r+1}\int_0^{+\infty} z^{(\omega-1/\gamma)r}h^{\omega r+1}(z)dz,$$

which yields

$$\begin{aligned} F \leq\; & 4B\left(\frac{r}{\gamma},1-2\frac{r}{\gamma}\right)B\left(2\frac{r}{\gamma},1-4\frac{r}{\gamma}\right)\gamma^{\omega r+1}\left(\omega+\frac{1}{r}\right)^{\omega r+1} \\ &\times\int_0^{+\infty} z^{(\omega-1/\gamma)r}h^{\omega r+1}(z)dz. \end{aligned} \tag{3.4}$$

Combining Equations (3.1), (3.2), (3.3), and (3.4), we find that

$$\begin{aligned}&\int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty}\frac{x^{2/\beta+1/\gamma-1}y^{2/\gamma+1/\alpha-1}z^{2/\alpha+1/\beta-1}}{|x-y-z|^{4/\alpha+4/\beta+4/\gamma}}\\&\quad\times F^{\eta+1/p}(x)G^{\theta+1/q}(y)H^{\omega+1/r}(z)dxdydz\\&\leq 4^{1/p}B^{1/p}\left(\frac{p}{\alpha},1-2\frac{p}{\alpha}\right)B^{1/p}\left(2\frac{p}{\alpha},1-4\frac{p}{\alpha}\right)\alpha^{\eta+1/p}\left(\eta+\frac{1}{p}\right)^{\eta+1/p}\\&\quad\times 2^{1/q}B^{1/q}\left(\frac{q}{\beta},\frac{q}{\beta}\right)B^{1/q}\left(2\frac{q}{\beta},1-4\frac{q}{\beta}\right)\beta^{\theta+1/q}\left(\theta+\frac{1}{q}\right)^{\theta+1/q}\\&\quad\times 4^{1/r}B^{1/r}\left(\frac{r}{\gamma},1-2\frac{r}{\gamma}\right)B^{1/r}\left(2\frac{r}{\gamma},1-4\frac{r}{\gamma}\right)\gamma^{\omega+1/r}\left(\omega+\frac{1}{r}\right)^{\omega+1/r}\\&\quad\times\left(\int_0^{+\infty}x^{(\eta-1/\alpha)p}f^{\eta p+1}(x)dx\right)^{1/p}\left(\int_0^{+\infty}y^{(\theta-1/\beta)q}g^{\theta q+1}(y)dy\right)^{1/q}\\&\quad\times\left(\int_0^{+\infty}z^{(\omega-1/\gamma)r}h^{\omega r+1}(z)dz\right)^{1/r}.\end{aligned}$$

This completes the proof. ■

Let us now discuss some notable cases of this result. By taking $\eta = 1/\alpha$, $\theta = 1/\beta$ and $\omega = 1/\gamma$, we obtain

$$\begin{aligned}&\int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty}\frac{x^{2/\beta+1/\gamma-1}y^{2/\gamma+1/\alpha-1}z^{2/\alpha+1/\beta-1}}{|x-y-z|^{4/\alpha+4/\beta+4/\gamma}}\\&\quad\times F^{1/\alpha+1/p}(x)G^{1/\beta+1/q}(y)H^{1/\gamma+1/r}(z)dxdydz\\&\leq 4^{1/p}B^{1/p}\left(\frac{p}{\alpha},1-2\frac{p}{\alpha}\right)B^{1/p}\left(2\frac{p}{\alpha},1-4\frac{p}{\alpha}\right)\alpha^{1/\alpha+1/p}\left(\frac{1}{\alpha}+\frac{1}{p}\right)^{1/\alpha+1/p}\\&\quad\times 2^{1/q}B^{1/q}\left(\frac{q}{\beta},\frac{q}{\beta}\right)B^{1/q}\left(2\frac{q}{\beta},1-4\frac{q}{\beta}\right)\beta^{1/\beta+1/q}\left(\frac{1}{\beta}+\frac{1}{q}\right)^{1/\beta+1/q}\\&\quad\times 4^{1/r}B^{1/r}\left(\frac{r}{\gamma},1-2\frac{r}{\gamma}\right)B^{1/r}\left(2\frac{r}{\gamma},1-4\frac{r}{\gamma}\right)\gamma^{1/\gamma+1/r}\left(\frac{1}{\gamma}+\frac{1}{r}\right)^{1/\gamma+1/r}\\&\quad\times\left(\int_0^{+\infty}f^{p/\alpha+1}(x)dx\right)^{1/p}\left(\int_0^{+\infty}g^{q/\beta+1}(y)dy\right)^{1/q}\left(\int_0^{+\infty}h^{r/\gamma+1}(z)dz\right)^{1/r},\end{aligned}$$

which corresponds to [7, Theorem 2]. Theorem 3.1 can thus be viewed as a three-parameter generalization.

By taking $\eta = 1/q + 1/r$, $\theta = 1/p + 1/r$ and $\omega = 1/p + 1/q$, we obtain the following elegant inequality:

$$\int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty} \frac{x^{2/\beta+1/\gamma-1}y^{2/\gamma+1/\alpha-1}z^{2/\alpha+1/\beta-1}}{|x-y-z|^{4/\alpha+4/\beta+4/\gamma}}F(x)G(y)H(z)dxdydz$$
$$\leq 4^{1/p}B^{1/p}\left(\frac{p}{\alpha},1-2\frac{p}{\alpha}\right)B^{1/p}\left(2\frac{p}{\alpha},1-4\frac{p}{\alpha}\right)\alpha$$
$$\times 2^{1/q}B^{1/q}\left(\frac{q}{\beta},\frac{q}{\beta}\right)B^{1/q}\left(2\frac{q}{\beta},1-4\frac{q}{\beta}\right)\beta$$
$$\times 4^{1/r}B^{1/r}\left(\frac{r}{\gamma},1-2\frac{r}{\gamma}\right)B^{1/r}\left(2\frac{r}{\gamma},1-4\frac{r}{\gamma}\right)\gamma$$
$$\times\left(\int_0^{+\infty}x^{(1-1/\alpha)p-1}f^p(x)dx\right)^{1/p}\left(\int_0^{+\infty}y^{(1-1/\beta)q-1}g^q(y)dy\right)^{1/q}$$
$$\times\left(\int_0^{+\infty}z^{(1-1/\gamma)r-1}h^r(z)dz\right)^{1/r}.$$

Note that F, G, and H are considered directly; they are raised to the power 1. To the best of the knowledge of the author, this integral inequality is new.

4. Conclusion

In this article, we present two new Hardy-Hilbert-type integral inequalities within a three-dimensional framework involving primitive functions. One inequality extends and generalizes [7, Theorem 2] by introducing additional parameters that enhance its flexibility and applicability. The distinct kernel function structures shed new light on the behavior of multi-variable integral operators. Future research could involve extending these inequalities to weighted settings, variable exponent spaces or fractional integral operators. Further investigations into their applications in functional analysis and partial differential equations also represent promising avenues for future study.

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