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Double butterfly theorem on ellipse

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Abstract The butterfly and the double butterfly theorem are applied to the circle. In this paper, we will modify the double butterfly theorem on an ellipse. The method of proving the double butterfly theorem on an ellipse can be done if Haruki's lemma can be applied to any ellipse. Thus, in this paper will also be given a modified Haruki's lemma on the ellipse.

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Keywords: Butterfly theorem, double butterfly theorem, Haruki's lemma

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1. Introduction

The butterfly theorem came from the butterfly problem, which is contained in the letter written by the astronomer Sir William Herschel to William Wallace [7]. Here is the content of Herschel's letter.

> I have kept a little problem for you which a friend of mine has sent me who says he cannot find a solution of it. I mentioned to him that I had a friend who would probably help him to one.
> The problem is this. Given AB the diameter of a circle.
> CD a chord cutting it at right angles in K
> EF, and HG two other chords drawn any how through the point K;
> and HF, EG chords joining the extrems of EF, HG.
> Required to prove that MK is equal to LK.

The following figure is the diagram attached by Herschel in his letter.

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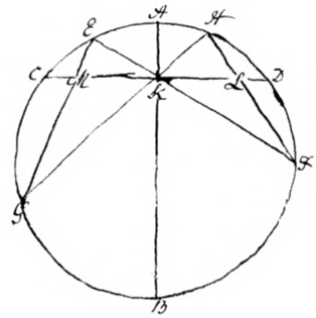

FIGURE 1. Diagram of butterfly problem from Herschel's letter

This problem seems to be of interest to some researchers, therefore there are several alternative solutions to the butterfly problem as in [4, 5, 8, 9]. Also, there is a modification of the butterfly problem by adding one butterfly in a circle, so that there are two butterflies in a circle. This modification is called the double butterfly theorem as in [10, 11]. Furthermore, there are comprehensive developments such as in [1, 3, 6].

The intersecting chords theorem applies to circle [12, p. 169], which will be used to prove both the butterfly theorem and the double butterfly theorem. However, the intersecting chords theorem does not apply to ellipses. Thus, another option that can be used to prove the double butterfly theorem on ellipses is Haruki's lemma on ellipses.

2. Double Butterfly Theorem

In the process of proving the double butterfly theorem on an ellipse, we need Haruki's lemma on the ellipse, so in this section, we will first introduce Haruki's lemma that applies to a circle, as given by [2].

Lemma 2.1. *(Haruki) Given two chords AB and CD of a circle that does not intersect, and the point P on arc AB that does not contain points C and D. Suppose E and F are the points of intersection of the chords PC with AB and PD with AB, respectively, as shown in Figure 2. The value of $(AE \cdot BF)/EF$ does not depend on the position of P.*

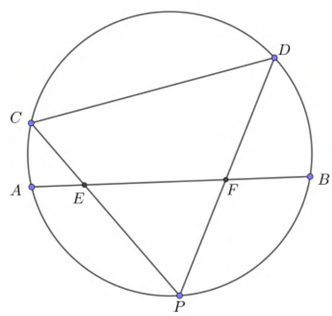

FIGURE 2. Haruki's lemma

Proof. The proof of Lemma 2.1 can be seen on [2]. ∎

The double butterfly theorem is basically constructing an equation that holds if two butterflies are constructed on a circle. The double butterfly theorem was first introduced by [11], and the various alternative proofs were discussed by [10]. The following theorem is the double butterfly theorem on a circle.

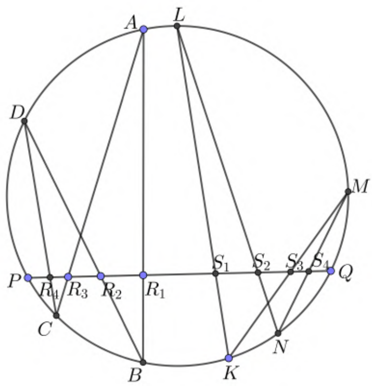

FIGURE 3. Double butterfly theorem

Theorem 2.2. *(Double Butterfly) Suppose PQ is any chord in a circle, butterfly $ABDC$ and $KLNM$ are on the circle such that the wings intersect PQ (in order from left to right) at R_4, R_3, R_2, R_1 and S_1, S_2, S_3, S_4, respectively. If $PR_1 = QS_1, PR_2 = QS_2$ and $PR_3 = QS_3$, then $PR_4 = QS_4$.*

Proof. As an illustration of the double butterfly theorem is shown in Figure 3, while the proof can be seen in [10, 11]. ∎

3. Double Butterfly Theorem on Ellipses

To prove the double butterfly theorem on an ellipse, it is necessary to modify Haruki's lemma on the ellipse. Haruki's lemma on ellipse has been discussed on [14], but in the proof, it uses analytic geometry while in this paper the proof of Haruki's lemma uses properties of euclidian geometry. The following is a modified form of Haruki's lemma on an ellipse.

Lemma 3.1. *(Haruki's Lemma on Ellipse) Given two chords AB and CD of an ellipse that does not intersect, the point P on arc AB does not contain the points C and D. Suppose that E and F are the points of intersection of chords PC with AB and PD with AB, respectively, as shown in Figure 4. The value of $(AE \cdot BF)/EF$ does not depend on the position of P.*

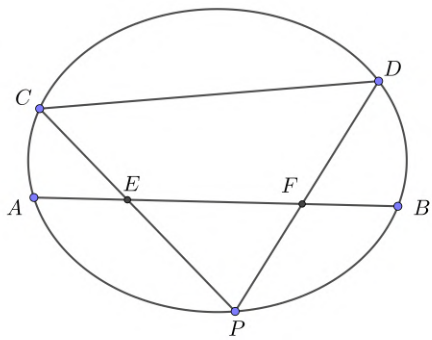

FIGURE 4. Haruki's lemma on ellipse

Proof. Construct a circle that passes through points A, P, and B and supposes that the extensions CP and DP intersect the circle at G and H, respectively, as shown in Figure 5. Then also construct a circle that passes through E, P, and H, and suppose that the extension of AB intersects the circle at I, as shown in Figure 6. Since both $\angle EPH$ and $\angle EIH$ are subtended by arc EH, then for any point P on arc AB we obtain $\angle EPH = \angle EIH$.

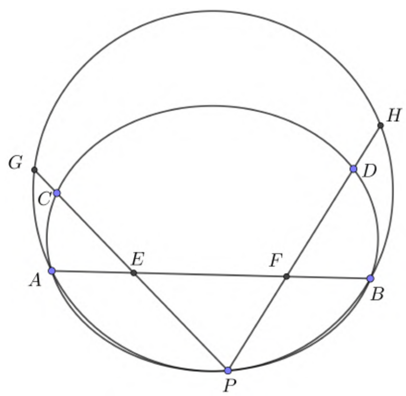

FIGURE 5. Construct circle that passes through point A, P, and B

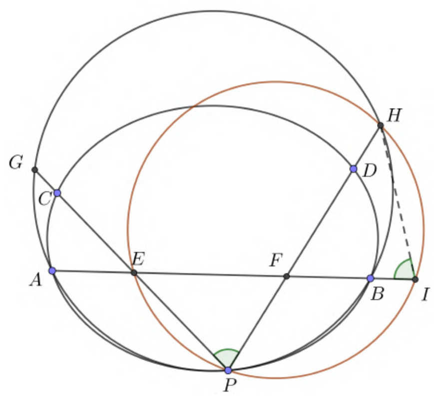

FIGURE 6. Construct circle that passes through point E, P, and H

Also, even if P is moved along the arc AB the point H will move, but the point I is always on the extension of the chord AB and does not move, as shown in Figure 7, because for any position of P we have $\angle EPH = \angle EIH$. So we have length BI is constant.

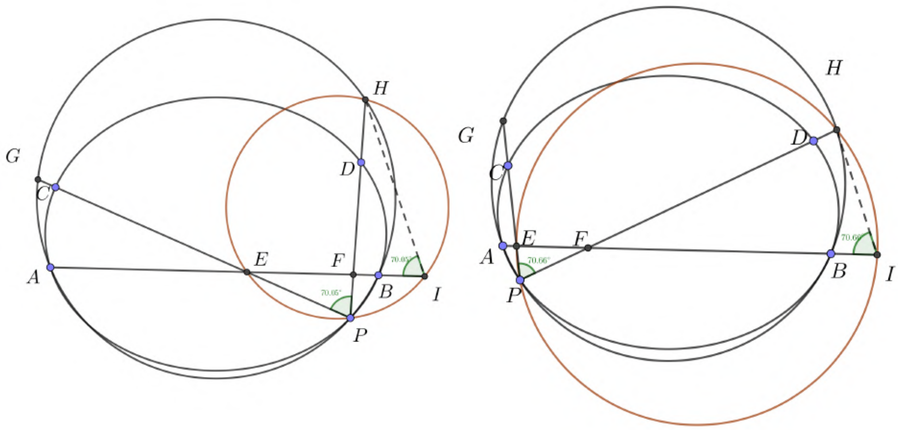

FIGURE 7. For any position of P, length BI does not change

Furthermore, applying the intersecting chords theorem, which has been proven in [13, p. 35], on the chord PH with AB and PH with EI we have

$$PF \cdot FH = AF \cdot FB, \tag{3.1}$$

and

$$PF \cdot FH = EF \cdot FI. \tag{3.2}$$

By equating the right-hand side of equation (3.1) with the right-hand side of equation (3.2) we get

$$\begin{aligned} AF \cdot FB &= EF \cdot FI, \\ (AE + EF)FB &= EF(FB + BI), \\ AE \cdot FB + AE \cdot FB &= EF \cdot FB + EF \cdot BI, \\ AE \cdot FB &= EF \cdot BI, \\ \frac{AE \cdot FB}{EF} &= BI. \end{aligned} \tag{3.3}$$

Since the length of BI is constant, the left-hand side of equation (3.3) is also constant. Thus Lemma 3.1 is proven. ∎

Theorem 3.2. *(Double Butterfly on Ellipse) Suppose PQ is any chord in an ellipse, butterfly $ABDC$ and $KLNM$ on an ellipse such that the wings intersect PQ (in order from left to right) at X_4, X_3, X_2, X_1 and Y_1, Y_2, Y_3, Y_4, respectively, as shown in Figure 8. If $PX_1 = QY_1, PX_2 = QY_2$ and $PX_3 = QY_3$, then $PX_4 = QY_4$.*

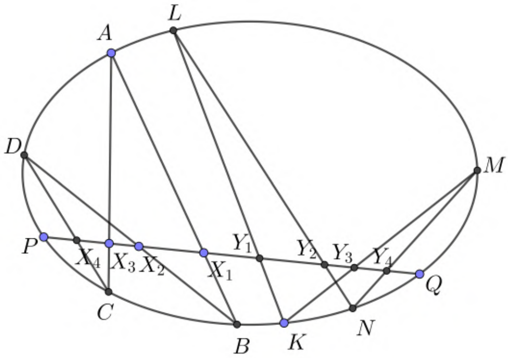

FIGURE 8. Double butterfly theorem on ellipse

Proof. To prove Theorem 3.2 we used Lemma 3.1. Consider the butterfly in Figure 8 as a group of chords, then each butterfly can be parsed as in Figure 9, Figure 10, Figure 11, and Figure 12. By observing Figure 9 and Figure 10 and because of Lemma 3.1 we get

$$\frac{PX_4 \cdot QX_3}{X_4X_3} = \frac{PX_2 \cdot QX_1}{X_2X_1}. \tag{3.4}$$

Also, by using the same argument as before at Figure 11 and Figure 12 we get

$$\frac{PY_3 \cdot QY_4}{Y_4Y_4} = \frac{PY_1 \cdot QY_2}{Y_1Y_2}. \tag{3.5}$$

Suppose

$$\left.\begin{aligned} PX_4 &= p, & X_2X_1 = Y_2Y_1 &= c,\\ QY_4 &= q, & X_1Y_1 &= d,\\ PX_3 = QY_3 &= a, & a+b+c+d &= e.\\ X_3X_2 = Y_3Y_2 &= b, \end{aligned}\right\} \tag{3.6}$$

Based on (3.6) and algebraic manipulation, equation (3.4) can be rewritten as

$$\frac{p(e+b+c)}{a-b} = \frac{(a+b)e}{c}, \tag{3.7}$$

and also equation (3.5) can be rewritten as

$$\frac{(e+b+c)q}{a-q} = \frac{e(a+b)}{c}. \tag{3.8}$$

By equating the left-hand side of equation (3.7) with the left-hand side of the equation (3.8), Theorem 3.2 is proven. ∎

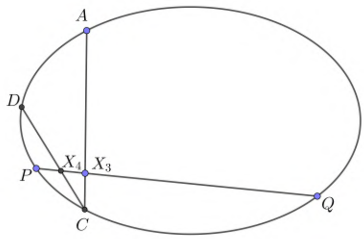

FIGURE 9. Chord DC and AC intersect PQ at X_4 and X_3

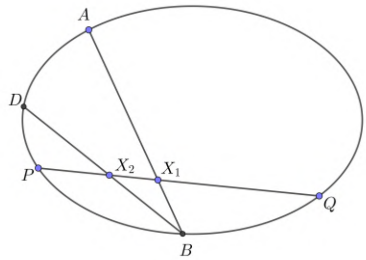

FIGURE 10. Chord DB and AB intersect PQ at X_2 and X_1

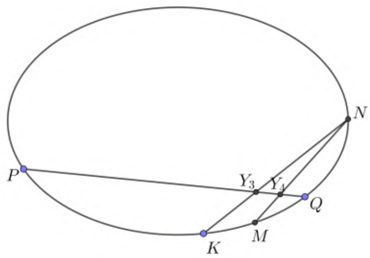

FIGURE 11. Chord KN and MN intersect PQ at Y_3 and Y_4

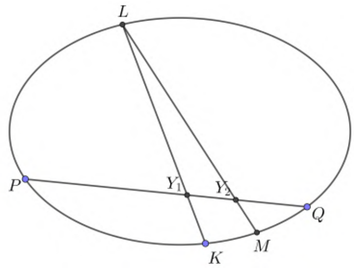

FIGURE 12. Chord KL and ML intersect PQ at Y_1 and Y_2

4. Concluding Remarks

Based on the results we obtained, it can be concluded that the double butterfly theorem which was originally applied to circle will still apply to the ellipse.

The method used in proving the double butterfly theorem on the ellipse is to observe that a butterfly is a group of chords. Then, by utilizing Haruki's lemma on the ellipse, it is proven that the double butterfly theorem still holds for the ellipse.

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