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A useful inequality for quaternion linear canonical transform

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Abstract In this work, we first introduce the quaternion Fourier transform. We explore its relation to the quaternion linear Fourier transform and utilize this fact to extend an inequality for the quaternion Fourier transform in the framework of the quaternion linear canonical transform.

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1. Introduction

As is known to all, the quaternion linear canonical transform (QLCT) is a left linear integral transformation with three extra parameters and has attracted much attention from several researchers in both theory and application. Several famous transformations such as the quaternion Fourier transform (QFT) and the quaternion fractional Fourier transform (QFrFT) are special forms of the OLCT. In recent years, many works have been demonstrated to transfer some essential properties of the quaternion Fourier transform to the quaternion linear canonical transform (see, e.g., [2, 9, 13–15]). In this regard, the authors [6] have proposed an uncertainty principle associated with the quaternion Fourier transform, which can be seen as the extension of the uncertainty principle for the classical Fourier transform. The present work aims to extend this uncertainty principle within the quaternion linear canonical transform. To achieve this goal, we first remind the definition of the quaternion linear canonical transform and then present the direct connection between this transformation and the quaternion Fourier transform and utilize this relation to obtain the inequality for the quaternion linear canonical transform.

The arrangement for this article is as below. Section 2 contains the basic facts related to quaternion algebra, that will be useful in the sequel. The definition of the quaternion Fourier transform (QFT) and essential properties are introduced in Section 3. Section 4 shortly recalls the definition of the quaternion linear canonical transform (QLCT) and its relation to the quaternion Fourier transform (QFT). Section 5 is devoted to deriving the

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inequality concerning the quaternion linear canonical transform (QLCT). A conclusion is made in the last section.

2. Quaternion Algebra with Properties

In this section, we present basic notations and results of quaternions that will be useful for this research. Let $\mathbb{H}$ be the associative algebra of real quaternion. Elements of quaternion algebra $\mathbb{H}$ can be represented as [7]

$$\mathbb{H} = \{q = q_0 + \mathbf{i}q_1 + \mathbf{j}q_2 + \mathbf{k}q_3 \mid q_0, q_1, q_2, q_3 \in \mathbb{R}\},$$

for which the three imaginary parts $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$ satisfy

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1. \tag{2.1}$$

Formula (2.1) tells us in general that quaternion multiplication is not commutative. For simplicity, denoted $q_0 = S(q)$ and $V(q) = \mathbf{i}\, q_1 + \mathbf{j}\, q_2 + \mathbf{k}\, q_3$ are the scalar and vector components of quaternion q, respectively. We may write any $q \in \mathbb{H}$ in the form

$$q = q_0 + \boldsymbol{q} = S(q) + V(q).$$

The quaternion conjugate $\overline{q}$ can be expressed in the form $\overline{q} = q_0 - \boldsymbol{q}$. This gives

$$\overline{qp} = \overline{p}\,\overline{q}, \quad \overline{\overline{p}} = p, \quad \overline{p+q} = \overline{p} + \overline{q}.$$

For every $q \in \mathbb{H}$, we may write the scalar and vector parts in the form

$$S(q) = \frac{q + \overline{q}}{2}, \quad \text{and} \quad V(q) = \frac{q - \overline{q}}{2}.$$

The module (norm) of quaternion q can be defined as

$$|q| = \sqrt{q\overline{q}} = \sqrt{q_0^2 + q_1^2 + q_2^2 + q_3^2}.$$

One can immediately check that for every $q, r, p \in \mathbb{H}$ it holds

$$S(q) \leq |q|, \quad |\boldsymbol{q}| = |\mathrm{V}(q)| \leq |q|, \quad \text{and} \quad S(qpr) = S(prq) = S(qpr).$$

Now we define the quaternion-valued inner product as

$$(f, g) = \int_{\mathbb{R}^2} f(\boldsymbol{x})\overline{g(\boldsymbol{x})}\, d\boldsymbol{x}, \quad d\boldsymbol{x} = dx_1 dx_2$$

with scalar product

$$\langle f, g\rangle = \int_{\mathbb{R}^2} S\big(f(\boldsymbol{x})\overline{g(\boldsymbol{x})}\big)\, d\boldsymbol{x} = S\bigg(\int_{\mathbb{R}^2} f(\boldsymbol{x})\overline{g(\boldsymbol{x})}\, d\boldsymbol{x}\bigg).$$

This yields the $L^2(\mathbb{R}^2; \mathbb{H})$-norm

$$\|f\|_{L^2(\mathbb{R}^2;\mathbb{H})} = \bigg(\int_{\mathbb{R}^2} |f(\boldsymbol{x})|^2\, d\boldsymbol{x}\bigg)^{1/2}.$$

3. Two-Sided Quaternion Fourier Transform

In this part, we start by reminding the definition of the two-sided quaternion Fourier transform (QFT). We present some inequalities related to this transformation, which will be needed later. A complete account of the QFT and its properties including the uncertainty principles, one can consult [1, 3–5, 8, 10–12].

Definition 3.1. The two-sided quaternion Fourier transform for a quaternion function f in $L^1(\mathbb{R}^2;\mathbb{H})$ is evaluated by

$$\mathcal{F}_H\{f\}(\boldsymbol{v}) = \int_{\mathbb{R}^2} e^{-\mathbf{i}2\pi v_1 x_1} f(\boldsymbol{x}) e^{-\mathbf{j}2\pi v_2 x_2}\, d\boldsymbol{x}, \tag{3.1}$$

where $\boldsymbol{x}, \boldsymbol{v} \in \mathbb{R}^2$.

Definition 3.2. For any $f \in L^1(\mathbb{R}^2;\mathbb{H})$ for which $\mathcal{F}_H\{f\} \in L^1(\mathbb{R}^2;\mathbb{H})$, its inverse is defined by

$$\mathcal{F}_H^{-1}\{\mathcal{F}_H\{f\}\}(\boldsymbol{x}) = f(\boldsymbol{x}) = \int_{\mathbb{R}^2} e^{\mathbf{i}2\pi v_1 x_1} \mathcal{F}_H\{f\}(\boldsymbol{v}) e^{\mathbf{j}2\pi v_2 x_2}\, d\boldsymbol{v}.$$

Using the decomposition of the quaternion function f, one may rewrite relation (3.1) in the form

$$\begin{aligned}\mathcal{F}_H\{f\}(\boldsymbol{v}) &= \int_{\mathbb{R}^2} e^{-\mathbf{i}2\pi v_1 x_1} \left(f_o(\boldsymbol{x}) + \mathbf{i} f_a(\boldsymbol{x}) + \mathbf{j} f_b(\boldsymbol{x}) + \mathbf{k} f_c(\boldsymbol{x})\right) e^{-\mathbf{j}2\pi v_2 x_2}\, d\boldsymbol{x} \\ &= \mathcal{F}_H\{f_o\}(\boldsymbol{v}) + \mathbf{i}\mathcal{F}_H\{f_a\}(\boldsymbol{v}) + \mathcal{F}_H\{f_b\}(\boldsymbol{v})\mathbf{j} + \mathbf{i}\mathcal{F}_H\{f_c\}(\boldsymbol{v})\mathbf{j},\end{aligned}$$

in which

$$\mathcal{F}_H\{f_i\}(\boldsymbol{v}) = \int_{\mathbb{R}^2} e^{-\mathbf{i}2\pi v_1 x_1} f_i(\boldsymbol{x}) e^{-\mathbf{j}2\pi v_2 x_2}\, d\boldsymbol{x}, \quad i = o, a, b, c.$$

Now we introduce the module of $\mathcal{F}_H\{f\}(\boldsymbol{v})$ as

$$|\mathcal{F}_H\{f\}(\boldsymbol{v})|_H^2 = |\mathcal{F}_H\{f_o\}(\boldsymbol{v})|^2 + |\mathcal{F}_H\{f_a\}(\boldsymbol{v})|^2 + |\mathcal{F}_H\{f_b\}(\boldsymbol{v})|^2 + |\mathcal{F}_H\{f_c\}(\boldsymbol{v})|^2.$$

Furthermore, we get the $L^p(\mathbb{R}^2;\mathbb{H})$-norm

$$\|\mathcal{F}_H\{f\}\|_{H,p} = \left(\int_{\mathbb{R}^2} |\mathcal{F}_H\{f\}(\boldsymbol{v})|_H^p\, d\boldsymbol{v}\right)^{1/p} \qquad f \in L^p(\mathbb{R}^2;\mathbb{H}).$$

Below we recall the component-wise uncertainty principle for the QFT in the following results.

Theorem 3.3. *For all $f \in L^1(\mathbb{R}^2;\mathbb{H}) \cap L^2(\mathbb{R}^2;\mathbb{H})$ such that $\frac{\partial}{x_k} f$ exists, there holds*

$$\int_{\mathbb{R}^2} x_k^2 |f(\boldsymbol{x})|^2\, d\boldsymbol{x} \int_{\mathbb{R}^2} v_k^2 |\mathcal{F}_H\{f\}(\boldsymbol{v})|_H^2\, d\boldsymbol{v} \geq \frac{1}{16\pi^2}\left(\int_{\mathbb{R}^2} |f(\boldsymbol{x})|^2\, d\boldsymbol{x}\right)^2, \quad k = 1, 2. \tag{3.2}$$

It is obvious that for $1 \leq p \leq 2$ we may change L^2-norm to L^p-norm on left-hand side of (3.2) and obtain the following result.

Theorem 3.4. *Under the conditions as above, one gets for $k = 1, 2$,*

$$\left(\int_{\mathbb{R}^2} x_k^p |f(\boldsymbol{x})|^p\, d\boldsymbol{x}\right)^{1/p} \left(\int_{\mathbb{R}^2} v_k^p |\mathcal{F}_H\{f\}(\boldsymbol{v})|_H^p\, d\boldsymbol{v}\right)^{1/p} \geq \frac{1}{4\pi} \int_{\mathbb{R}^2} |f(\boldsymbol{x})|^2\, d\boldsymbol{x}. \tag{3.3}$$

In particular,

$$\left(\int_{\mathbb{R}^2} x_k^p |f(\boldsymbol{x})|^p \, d\boldsymbol{x}\right)^{1/p} \left(\int_{\mathbb{R}^2} |\mathcal{F}_H\{f\}(\frac{\boldsymbol{v}}{2\pi})|_H^p \, d\boldsymbol{v}\right)^{1/p} \geq \frac{(2\pi)^{1+\frac{2}{p}}}{4\pi} \int_{\mathbb{R}^2} |f(\boldsymbol{x})|^2 \, d\boldsymbol{x}.$$

4. Quaternion Linear Canonical Transform (QLCT)

In this section, we recall the definition of the two-sided quaternion linear canonical transform (QLCT) and study its connection to the quaternion Fourier transform (QFT). More details have presented in [2, 13–15].

Definition 4.1. Let $M_1 = (a_1, b_1, c_1, d_1) = \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} \in \mathbb{R}^{2\times 2}$ and $M_2 = (a_2, b_2, c_2, d_2) = \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix} \in \mathbb{R}^{2\times 2}$ be matrix parameters such that $\det(M_1) = \det(M_1) = 1$. The two-sided quaternion linear canonical transform of a quaternion function $f \in L^2(\mathbb{R}^2; \mathbb{H})$ is defined through

$$\mathcal{L}^Q_{M_1,M_2}\{f\}(\boldsymbol{v}) = \begin{cases} \int_{\mathbb{R}^2} K_{M_1}(x_1, v_1) f(\boldsymbol{x}) K_{M_2}(x_2, v_2)\, d\boldsymbol{x}, \text{ for } b_1 b_2 \neq 0 \\ \sqrt{d_1}\, e^{\mathbf{i}\left(\frac{c_1 d_1}{2}\right) v_1^2} f(d_1 v_1, d_2 v_2) \sqrt{d_2}\, e^{\mathbf{j}\left(\frac{c_2 d_2}{2}\right) v_2^2}, \text{ for } b_1 b_2 = 0, \end{cases} \tag{4.1}$$

where

$$K_{M_1}(x_1, v_1) = \frac{1}{\sqrt{2\pi b_1}}\, e^{\frac{\mathbf{i}}{2}\left(\frac{a_1}{b_1}x_1^2 - \frac{2}{b_1}x_1 v_1 + \frac{d_1}{b_1}v_1^2 - \frac{\pi}{2}\right)},$$

and

$$K_{M_2}(x_2, v_2) = \frac{1}{\sqrt{2\pi b_2}}\, e^{\frac{\mathbf{j}}{2}\left(\frac{a_2}{b_2}x_2^2 - \frac{2}{b_2}x_2 v_2 + \frac{d_2}{b_2}v_2^2 - \frac{\pi}{2}\right)}.$$

denote the QLCT kernel functions.

In this work, we consider Definition 4.1 in the case of $b_1 b_2 \neq 0$. Especially, when $M_1 = M_2 = (a_i, b_i, c_i, d_i) = (0, 1, -1, 0)$ with $i = 1, 2$, the two-sided QLCT definition (4.1) boils down to the two-sided QFT definition:

$$\mathcal{L}^Q_{M_1,M_2}\{f\}(\boldsymbol{v}) = \int_{\mathbb{R}^2} \frac{e^{-\mathbf{i}\frac{\pi}{4}}}{\sqrt{2\pi}} e^{-\mathbf{i}v_1 x_1}\, f(\boldsymbol{x}) e^{-\mathbf{j}v_2 x_2} \frac{e^{-\mathbf{j}\frac{\pi}{4}}}{\sqrt{2\pi}}\, d\boldsymbol{x} = \frac{e^{-\mathbf{i}\frac{\pi}{4}}}{\sqrt{2\pi}} \mathcal{F}_H\{f\}\left(\frac{\boldsymbol{v}}{2\pi}\right) \frac{e^{-\mathbf{j}\frac{\pi}{4}}}{\sqrt{2\pi}}. \tag{4.2}$$

where $\mathcal{F}_H\{f\}$ is defined by (3.1). From equation (4.2), we infer that

$$2\pi \left|\mathcal{L}^Q_{M_1,M_2}\{f\}(\boldsymbol{v})\right| = \left|\mathcal{F}_H\{f\}\left(\frac{\boldsymbol{v}}{2\pi}\right)\right|.$$

The inverse transform of the two-sided QLCT (4.1) above is computed by

$$\begin{aligned} f(\boldsymbol{x}) &= \left(\mathcal{L}^Q_{M_1,M_2}\right)^{-1}\left[\mathcal{L}^Q_{M_1,M_2}\{f\}\right](\boldsymbol{x}) \\ &= \frac{1}{2\pi\sqrt{b_1 b_2}} \int_{\mathbb{R}^2} e^{-\frac{\mathbf{i}}{2}(m_1)} \mathcal{L}^Q_{M_1,M_2}\{f\}(\boldsymbol{v}) e^{-\frac{\mathbf{j}}{2}(m_2)}\, d\boldsymbol{v}, \end{aligned}$$

where $m_i = \frac{a_i}{b_i}x_i^2 - \frac{2}{b_i}x_i v_i + \frac{d_i}{b_i}v_i^2 - \frac{\pi}{2}$, $i = 1, 2$, provided that the integral exists.

By virtue of (4.1), we have

$$\begin{aligned} &\mathcal{L}^Q_{M_1,M_2}\{f\}(\boldsymbol{v}) \\ &= \int_{\mathbb{R}^2} \frac{e^{-\mathbf{i}\frac{\pi}{4}}}{\sqrt{2\pi b_1}} e^{\mathbf{i}\frac{d_1}{2b_1}v_1^2} e^{-\mathbf{i}\frac{x_1 v_1}{b_1}} e^{\mathbf{i}\frac{a_1}{2b_1}x_1^2} f(\boldsymbol{x}) \frac{e^{-\mathbf{j}\frac{\pi}{4}}}{\sqrt{2\pi b_2}} e^{\mathbf{j}\frac{d_2}{2b_2}v_2^2} e^{-\mathbf{j}\frac{x_2 v_2}{b_2}} e^{\mathbf{j}\frac{a_2}{2b_2}x_2^2}\, d\boldsymbol{x}. \end{aligned}$$

It is straightforward to verify that the connection between the two-sided QFT and the two-sided QLCT is given by

$$\begin{aligned}\sqrt{2\pi b_1}\, e^{\mathbf{i}\frac{\pi}{4}} e^{-\mathbf{i}\frac{d_1}{2b_1}v_1^2}\mathcal{L}^Q_{M_1,M_2}\{f\}(\boldsymbol{v})e^{-\mathbf{j}\frac{d_2}{2b_2}v_2^2}\sqrt{2\pi b_2}\, e^{\mathbf{j}\frac{\pi}{4}} &= \mathcal{F}_H\{\check{f}\}\left(\frac{v_1}{b_1},\frac{v_2}{b_2}\right)\\ &= \mathcal{F}_H\{\check{f}\}\left(\frac{\boldsymbol{v}}{\boldsymbol{b}}\right), \end{aligned} \tag{4.3}$$

where

$$\check{f}(\boldsymbol{x}) = e^{\mathbf{i}\frac{a_1}{2b_1}x_1^2} f(\boldsymbol{x}) e^{\mathbf{j}\frac{a_2}{2b_2}x_2^2}. \tag{4.4}$$

For every $f \in L^2(\mathbb{R}^2;\mathbb{H})$, there holds

$$\int_{\mathbb{R}^2}\left|\mathcal{L}^Q_{M_1,M_2}\{f\}(\boldsymbol{v})\right|^2 d\boldsymbol{v} = \int_{\mathbb{R}^2}|f(\boldsymbol{x})|^2\, d\boldsymbol{x},$$

which is usually called Parseval's theorem for the two-sided QLCT.

5. Inequality for QLCT

Now we are in a position to prove inequalities concerning the QLCT, which are the main results of this paper.

Theorem 5.1. *Under the same assumptions as in Theorem 3.3, one has*

$$\begin{aligned}\left(\int_{\mathbb{R}^2} x_k^p |f(\boldsymbol{x})|^p\, d\boldsymbol{x}\right)^{1/p}\left(\int_{\mathbb{R}^2} v_k^p \left|\mathcal{L}^Q_{M_1,M_2}\{f\}(\boldsymbol{v})\right|_H^p d\boldsymbol{v}\right)^{1/p}&\\ \geq \frac{b_k|b_1b_2|^{\frac{1}{p}-\frac{1}{2}}}{8\pi^2}\int_{\mathbb{R}^2}|f(\boldsymbol{x})|^2\, d\boldsymbol{x}, \quad k=1,2.&\end{aligned} \tag{5.1}$$

Proof. Upon replacing $f(\boldsymbol{x})$ by $\check{f}(\boldsymbol{x})$ as in (4.4) on both sides of (3.3), we get

$$\left(\int_{\mathbb{R}^2} x_k^p |\check{f}(\boldsymbol{x})|^p\, d\boldsymbol{x}\right)^{1/p}\left(\int_{\mathbb{R}^2} v_k^p |\mathcal{F}_H\{\check{f}\}(\boldsymbol{v})|_H^p\, d\boldsymbol{v}\right)^{1/p} \geq \frac{1}{4\pi}\int_{\mathbb{R}^2}|\check{f}(\boldsymbol{x})|^2\, d\boldsymbol{x}. \tag{5.2}$$

Setting $\boldsymbol{v} = \frac{\boldsymbol{v}}{\boldsymbol{b}}$, we have

$$\left(\int_{\mathbb{R}^2} x_k^p |\check{f}(\boldsymbol{x})|^p\, d\boldsymbol{x}\right)^{1/p}\left(\int_{\mathbb{R}^2} \frac{v_k^p}{b_k^p}\frac{1}{|b_1b_2|}\left|\mathcal{F}_H\{\check{f}\}\left(\frac{\boldsymbol{v}}{\boldsymbol{b}}\right)\right|_H^p d\boldsymbol{v}\right)^{1/p} \geq \frac{1}{4\pi}\int_{\mathbb{R}^2}|\check{f}(\boldsymbol{x})|^2\, d\boldsymbol{x}. \tag{5.3}$$

By inserting (4.3) and (4.4) into equation (5.3) we obtain

$$\begin{aligned}&\left(\int_{\mathbb{R}^2} x_k^p \left|e^{\mathbf{i}\frac{a_1}{2b_1}x_1^2} f(\boldsymbol{x}) e^{\mathbf{j}\frac{a_2}{2b_2}x_2^2}\right|^p d\boldsymbol{x}\right)^{1/p}\left(\int_{\mathbb{R}^2}\frac{v_k^p}{b_k^p}\frac{1}{|b_1b_2|}\left|\sqrt{2\pi b_1}\, e^{\mathbf{i}\frac{\pi}{4}} e^{-\mathbf{i}\frac{d_1}{2b_1}v_1^2}\right.\right.\\ &\left.\left.\times\, \mathcal{L}^Q_{M_1,M_2}\{f\}(\boldsymbol{v})e^{-\mathbf{j}\frac{d_2}{2b_2}v_2^2}\sqrt{2\pi b_2}\, e^{\mathbf{j}\frac{\pi}{4}}\right|_H^p d\boldsymbol{v}\right)^{1/p} \geq \frac{1}{4\pi}\int_{\mathbb{R}^2}\left|e^{\mathbf{i}\frac{a_1}{2b_1}x_1^2} f(\boldsymbol{x}) e^{\mathbf{j}\frac{a_2}{2b_2}x_2^2}\right|^2 d\boldsymbol{x}.\end{aligned} \tag{5.4}$$

Simplifying it yields

$$\left(\int_{\mathbb{R}^2} x_k^p |f(\boldsymbol{x})|^p\, d\boldsymbol{x}\right)^{1/p} \left(\int_{\mathbb{R}^2} \frac{v_k^p}{b_k^p} |b_1 b_2|^{\frac{p}{2}-1} \left|2\pi \mathcal{L}^Q_{A_1,M_2}\{f\}(\boldsymbol{v})\right|_H^p d\boldsymbol{v}\right)^{1/p} \geq \frac{1}{4\pi}\int_{\mathbb{R}^2} |f(\boldsymbol{x})|^2\, d\boldsymbol{x}.$$

We further obtain

$$\left(\int_{\mathbb{R}^2} x_k^p |f(\boldsymbol{x})|^p\, d\boldsymbol{x}\right)^{1/p} |b_1 b_2|^{\frac{1}{2}-\frac{1}{p}} \left(\int_{\mathbb{R}^2} v_k^p \left|2\pi \mathcal{L}^Q_{M_1,M_2}\{f\}(\boldsymbol{v})\right|_H^p d\boldsymbol{v}\right)^{1/p} \geq \frac{b_k}{4\pi}\int_{\mathbb{R}^2} |f(\boldsymbol{x})|^2\, d\boldsymbol{x},$$

which is the same as

$$\left(\int_{\mathbb{R}^2} x_k^p |f(\boldsymbol{x})|^p\, d\boldsymbol{x}\right)^{1/p} \left(\int_{\mathbb{R}^2} v_k^p \left|\mathcal{L}^Q_{M_1,M_2}\{f\}(\boldsymbol{v})\right|_H^p d\boldsymbol{v}\right)^{1/p} \geq \frac{b_k |b_1 b_2|^{\frac{1}{p}-\frac{1}{2}}}{8\pi^2}\int_{\mathbb{R}^2} |f(\boldsymbol{x})|^2\, d\boldsymbol{x},$$

and the proof is complete. ∎

Theorem 5.2. *If $f \in L^1(\mathbb{R}^2;\mathbb{H}) \cap L^2(\mathbb{R}^2;\mathbb{H})$ and $\mathcal{L}^Q_{M_1,M_2}\{f\}$ exists and is also in $L^2(\mathbb{R}^2;\mathbb{H})$, then*

$$\left(\int_{\mathbb{R}^2} (x_1^p + x_2^p)|f(\boldsymbol{x})|^p\, d\boldsymbol{x}\right)^{1/p} \left(\int_{\mathbb{R}^2} \left(\frac{v_1^p}{b_1^p} + \frac{v_2^p}{b_2^p}\right) \left|\mathcal{L}^Q_{M_1,M_2}\{f\}(\boldsymbol{v})\right|_H^p d\boldsymbol{v}\right)^{1/p} \geq \frac{2^{-2p+1}}{\pi^{p+1}} |b_1 b_2|^{\frac{1}{p}-\frac{1}{2}} \left(\int_{\mathbb{R}^2} |f(\boldsymbol{x})|^2\, d\boldsymbol{x}\right)^p,$$

for $1 \leq p \leq 2$.

Proof. Using the procedure as in equations (5.2), (5.3), and (5.4), we obtain

$$\left(\int_{\mathbb{R}^2} (x_1^p + x_2^p)|f(\boldsymbol{x})|^p\, d\boldsymbol{x}\right)^{1/p} \left(|b_1 b_2|^{-1}\int_{\mathbb{R}^2} \left(\frac{v_1^p}{b_1^p} + \frac{v_2^p}{b_2^p}\right) \left|\mathcal{F}_H\{\check{f}\}(\frac{\boldsymbol{v}}{\boldsymbol{b}})\right|_H^p d\boldsymbol{v}\right)^{1/p} \geq \frac{4}{(4\pi)^p}\left(\int_{\mathbb{R}^2} |f(\boldsymbol{x})|^2\, d\boldsymbol{x}\right)^p.$$

Consequently,

$$\left(\int_{\mathbb{R}^2} (x_1^p + x_2^p)|f(\boldsymbol{x})|^p\, d\boldsymbol{x}\right)^{1/p} |b_1 b_2|^{\frac{1}{2}-\frac{1}{p}} \left(\int_{\mathbb{R}^2} \left(\frac{v_1^p}{b_1^p} + \frac{v_2^p}{b_2^p}\right) \left|2\pi\mathcal{L}^Q_{M_1,M_2}\{f\}(\boldsymbol{v})\right|_H^p d\boldsymbol{v}\right)^{1/p} \geq \frac{4}{(4\pi)^p}\left(\int_{\mathbb{R}^2} |f(\boldsymbol{x})|^2\, d\boldsymbol{x}\right)^p.$$

This yields

$$\left(\int_{\mathbb{R}^2} (x_1^p + x_2^p)|f(\boldsymbol{x})|^p\, d\boldsymbol{x}\right)^{1/p} \left(\int_{\mathbb{R}^2} \left(\frac{v_1^p}{b_1^p} + \frac{v_2^p}{b_2^p}\right) \left|\mathcal{L}^Q_{M_1,M_2}\{f\}(\boldsymbol{v})\right|_H^p d\boldsymbol{v}\right)^{1/p} \geq \frac{2^{-2p+1}}{\pi^{p+1}} |b_1 b_2|^{\frac{1}{p}-\frac{1}{2}} \left(\int_{\mathbb{R}^2} |f(\boldsymbol{x})|^2\, d\boldsymbol{x}\right)^p.$$

The proof is complete. ■

Remark 5.3. It should be noticed that in the case of $M_1 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ and $M_2 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, equation (5.1) turns into equation (3.3).

6. Conclusion

In this paper, we have introduced the quaternion Fourier transform and the quaternion linear canonical transform. We have presented an inequality related to the quaternion Fourier transform and generalized within the quaternion linear canonical transform.

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References

[1] A. Abouelaz, A. Achak, R. Daher, N. Safouane, Donoho-Stark's uncertainty principle for the quaternion Fourier transform. *Bol. Soc. Mat. Mex.* **26** (2020) 587–597.

[2] A. Achak, A. Abouelaz, R. Daher, N. Safouane, Uncertainty principles for the quaternion linear canonical transform, *Adv. App. Clifford Algebr.* **29** (2019) 99.

[3] M. Bahri, M. Saleh, Relation between quaternion Fourier transform and quaternion Wigner-Ville distribution associated with linear canonical transform, *J. Appl. Math.* **2017** (2017) 3247364. https://doi.org/10.1155/2017/3247364.

[4] M. Bahri, Quaternion algebra-valued wavelet transform, *Appl. Math. Sci.* **5** (2011) 3531-3540.

[5] M. Bahri, A. Lawi, N. Aris, M. Saleh, M. Nur, Relationships between convolution and correlation for Fourier transform and quaternion Fourier transform, *Int. J. Math. Anal.* **7** (2013) 2101-2109.

[6] M. Bahri, S. A. Karim, A variation on inequality for quaternion Fourier transform, modified convolution and correlation theorems for general quaternion linear canonical transform, *Symmetry* **14** (2022) 1303. https://doi.org/10.3390/sym14071303.

[7] T. Bülow, Hypercomplex Spectral Signal Representations for the Processing and Analysis of Images,*Ph.D. thesis*, University of Kiel, Germany (1999).

[8] L. P. Chen, K. I. Kou, M. S. Liu, Pitt's inequality and the uncertainty principle associated with the quaternion Fourier transform, *J. Math. Anal. Appl.* **423** (2015) 681–700.

[9] W. B. Gao, B. Z. Li, Quaternion windowed linear canonical transform of two-dimensional signals, *Adv. Appl. Clifford Algebr.* **30** (2020) 16. https://doi.org/10.1007/s00006-020-1042-4

[10] E. Hitzer, Quaternion Fourier transform on quaternion fields and generalizations, *Adv. App. Clifford Algebr.* **20** (2007) 497–517.

[11] P. Lian, Uncertainty principle for the quaternion Fourier transform, *J. Math. Anal. Appl.* **467** (2018) 1258–1269.

[12] P. Lian, Sharp Hausdorff-Young inequalities for the quaternion Fourier transforms, *Proc. Am. Math. Soc.* **148** (2020) 697–703.

[13] Y. N. Zhang, B. Z. Li, Novel uncertainty principles for two-sided quaternion linear canonical transform, *Adv. App. Clifford Algebr.* **28** (2018) 15. https://doi.org/10.1007.

[14] X. Zhu, S. Zheng, Uncertainty principles for the two-sided quaternion linear canonical transform, *Circuits, Syst. Signal Process.* **39** (2020) 4436–4458.

[15] X. Zhu, S. Zheng, On uncertainty principle for the two-sided quaternion linear canonical transform, *J. Pseudo-Differ. Oper. Appl.* **12** (2021). https://doi.org/10.1007/s11868-021-00395-x.