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A Note on Euclidean Spaces $\mathbb{R}^n$ and n-Normed Spaces

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Abstract This article gives us a relation between Euclidean space $\mathbb{R}^n$ and a subspace of X (an n-normed space) using properties of the determinant of square matrices. We can calculate the n-norm of n vectors in the subspace in a simpler way, from the product of a constant and an n-norm of other n vectors. Some functionals will be investigated in these spaces. Furthermore, we also define a norm induced from an inner product on the subspace.

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1. Introduction

Many functionals can be well defined on $\mathbb{R}^n$. Usually, at the beginning of a study about vector spaces, we use $\mathbb{R}^n$, cause especially for $n = 2$ or $n = 3$, the visual graphic and the geometric interpretation of these spaces are still relatively easy. In vector spaces, we will discuss one of the interesting functionals. By [6], we recall the definition of an n-norm on a real vector space X, with $\dim(X) \geq n$. It is a mapping $\|\cdot, \dots, \cdot\| : X \times \cdots \times X \longrightarrow \mathbb{R}$ which satisfies the following four conditions:

(1) $\|x_1, \cdots, x_n\| \geq 0$ holds for every $x_1, \dots, x_n \in X$;
$\|x_1, \cdots, x_n\| = 0$ if and only if $x_1, \dots, x_n$ are linearly dependent;
(2) $\|x_1, \cdots, x_n\|$ is invariant under permutation;
(3) $\|\alpha x_1, \cdots, x_n\| = |\alpha| \|x_1, \cdots, x_n\|$ holds for every $x_1, \cdots, x_n \in X$ and for every scalar $\alpha \in \mathbb{R}$;
(4) $\|x_1, \cdots, x_{n-1}, y + z\| \leq \|x_1, \cdots, x_{n-1}, y\| + \|x_1, \cdots, x_{n-1}, z\|$ holds for every $x_1, x_2, \dots, x_{n-1}, y, z \in X$.

Now we call that the pair $(X, \|\cdot, \cdots, \cdot\|)$ is an n-normed space. We have known that the geometrical interpretation of the n-norm is the volume of the n-dimensional parallelepiped spanned by n elements of vector spaces. The development of the theory of n-normed spaces, with $n = 2$, was started in the late 1960s. Gähler had an idea to generalize an area in a real vector space. He was an initiator and first introduced it (see [3, 4]). We also see some recent results in [6, 9, 10].

In this article, we will investigate the relation between a subspace of X (as n-normed space) and $\mathbb{R}^n$. We will use the n-norm properties optimally. In addition, knowledge of square matrices and its determinant also plays an important role.

2. Main Results

Here, we give an n-normed space $X = (X, \|\cdot, \cdots, \cdot\|)$. Next, suppose that a fixed n linearly independent vectors, namely

$$G = \{g_1, \cdots, g_n\} \subset X. \tag{2.1}$$

Now, one may take a subspace of X as follows

$$Y := \text{span}(G). \tag{2.2}$$

By taking $h \in Y$, we can find $c_h = (c_{1h}, \cdots, c_{nh}) \in \mathbb{R}^n$ such that $h = c_{1h}g_1 + \cdots + c_{nh}g_n$ holds. See again n-normed space X and based on the properties of n-norm, we get the following property.

Lemma 2.1. *Let X be an n-normed space. If $f_1, f_2, \cdots, f_n \in X$, then we have*

$$\|f_1 + \alpha f_i, f_2, \cdots, f_n\| = \|f_1, f_2, \cdots, f_n\|$$

for every $i = 2, 3, \cdots, n$ and for every $\alpha \in \mathbb{R}$.

Proof. Suppose that X is an n-normed space. Take an arbitrary $f_1, f_2, \cdots, f_n \in X$ and $\alpha \in \mathbb{R}$. By triangle inequality of n-normed and homogenity

$$\|f_1 + \alpha f_i, f_2, \cdots, f_n\| \leq \|f_1, f_2, \cdots, f_n\| + |\alpha| \|f_i, f_2, \cdots, f_n\|$$

where $i \in \{2, 3, \cdots, n\}$. Consequently, $\|f_i, f_2, \cdots, f_n\| = 0$, so

$$\|f_1 + \alpha f_i, f_2, \cdots, f_n\| \leq \|f_1, f_2, \cdots, f_n\|.$$

We also obtain

$$\begin{aligned} \|f_1, f_2, \cdots, f_n\| &= \|(f_1 + \alpha f_i) - \alpha f_i, f_2, \cdots, f_n\| \\ &\leq \|f_1 + \alpha f_i, f_2, \cdots, f_n\| + |\alpha| \|f_i, f_2, \cdots, f_n\| \\ &= \|f_1 + \alpha f_i, f_2, \cdots, f_n\|. \end{aligned}$$

Hence $\|f_1 + \alpha f_i, f_2, \cdots, f_n\| = \|f_1, f_2, \cdots, f_n\|$. ∎

Lemma 2.1 is equivalent to

$$\left\| f_1 + \sum_{i=2}^{n} f_i, f_2, \cdots, f_n \right\| = \|f_1, f_2, \cdots, f_n\|. \tag{2.3}$$

In fact, this form is used in proving the Proposition 2.2 and some propositions below directly. We can see this one as a corollary of Lemma 2.1.

2.1. Scalar by The Determinant of A Square Matrix

In this subsection, we work step by step to obtain the general formula. Note that for $b \in \mathbb{R}$, $|b|$ means **absolute value** of b. Meanwhile, for B_n be real square matrix with $n \geq 2$, $|B_n|$ means **determinant value of** B_n. So, we use $\text{abs}(|B_n|)$ to say the absolute value of the determinant of B_n. Now, we start with $n = 2$ in the following proposition.

Proposition 2.2. *Let an n-normed space X, (2.1) and (2.2). If $p_1, p_2 \in Y$ with*

$$p_1 = \sum_{i=1}^{n} c_{ip_1} g_i \quad \textit{and} \quad p_2 = \sum_{i=1}^{n} c_{ip_2} g_i,$$

then we have $\|p_1, p_2, g_3, \cdots, g_n\| = \text{abs}\left(\begin{vmatrix} c_{1p_1} & c_{2p_1} \\ c_{1p_2} & c_{2p_2} \end{vmatrix}\right) \|g_1, g_2, g_3, \cdots, g_n\|.$

Proof. We consider the above assumptions. Check that p_1 and p_2 have

$$c_{p_1} = \begin{bmatrix} c_{1p_1} & c_{2p_1} & \cdots & c_{np_1} \end{bmatrix}, c_{p_2} = \begin{bmatrix} c_{1p_2} & c_{2p_2} & \cdots & c_{np_2} \end{bmatrix} \in \mathbb{R}^n$$

such that

$$p_1 = \sum_{i=1}^{n} c_{ip_1} g_i \quad \text{and} \quad p_2 = \sum_{i=1}^{n} c_{ip_2} g_i.$$

Using Lemma 2.1, check the following

$$\begin{aligned} \|p_1, p_2, g_3, \cdots, g_n\| &= \left\| \sum_{i=1}^{n} c_{ip_1} g_i, \sum_{i=1}^{n} c_{ip_2} g_i, g_3, \cdots, g_n \right\| \\ &= \|c_{1p_1} g_1 + c_{2p_1} g_2, c_{1p_2} g_1 + c_{2p_2} g_2, g_3, \cdots, g_n\|. \end{aligned} \tag{2.4}$$

See that we obtain $\begin{bmatrix} c_{1p_1} & c_{2p_1} \end{bmatrix}$ and $\begin{bmatrix} c_{1p_2} & c_{2p_2} \end{bmatrix}$ in $\mathbb{R}^2$ by

$$c_{1p_1} g_1 + c_{2p_1} g_2 \text{ and } c_{1p_2} g_1 + c_{2p_2} g_2.$$

We have two cases. First case: for $c_{1p_1} g_1 + c_{2p_1} g_2$ and $c_{1p_2} g_1 + c_{2p_2} g_2$ be linearly dependent, we get linearly dependent vectors $\begin{bmatrix} c_{1p_1} & c_{2p_1} \end{bmatrix}$ and $\begin{bmatrix} c_{1p_2} & c_{2p_2} \end{bmatrix}$ in $\mathbb{R}^2$. Now we have

$$\begin{vmatrix} c_{1p_1} & c_{2p_1} \\ c_{1p_2} & c_{2p_2} \end{vmatrix} = 0 \quad \text{and} \quad \|c_{1p_1} g_1 + c_{2p_1} g_2, c_{1p_2} g_1 + c_{2p_2} g_2, g_3, \cdots, g_n\| = 0.$$

Consequently, $\|p_1, p_2, g_3, \cdots, g_n\| = 0$ and $\text{abs}\left(\begin{vmatrix} c_{1p_1} & c_{2p_1} \\ c_{1p_2} & c_{2p_2} \end{vmatrix}\right) \|g_1, g_2, g_3, \cdots, g_n\| = 0.$

Second case: for $\begin{bmatrix} c_{1p_1} & c_{2p_1} \end{bmatrix}$ and $\begin{bmatrix} c_{1p_2} & c_{2p_2} \end{bmatrix}$ be linearly independent in $\mathbb{R}^2$. It can be checked that $\begin{vmatrix} c_{1p_1} & c_{2p_1} \\ c_{1p_2} & c_{2p_2} \end{vmatrix} \neq 0$ holds. Since $c_{1p_1} g_1 + c_{2p_1} g_2, c_{1p_2} g_1 + c_{2p_2} g_2, g_3, \cdots, g_{n-1}$, and g_n are linearly independent, then $\|c_{1p_1} g_1 + c_{2p_1} g_2, c_{1p_2} g_1 + c_{2p_2} g_2, g_3, \cdots, g_n\| \neq 0$. Here without losing generality, let $c_{1p_1} \neq 0$ and define

$$\mathcal{K} := \|c_{1p_1} g_1 + c_{2p_1} g_2, c_{1p_2} g_1 + c_{2p_2} g_2, g_3, \cdots, g_n\|.$$

Next, we multiply with $|c_{1p_1}|$

$$\begin{aligned} |c_{1p_1}|\mathcal{K} &= \|c_{1p_1} g_1 + c_{2p_1} g_2, c_{1p_1}(c_{1p_2} g_1 + c_{2p_2} g_2), g_3, \cdots, g_n\| \\ &= \|c_{1p_1} g_1 + c_{2p_1} g_2, c_{1p_2}(c_{1p_1} g_1 + c_{2p_1} g_2) + (c_{1p_1} c_{2p_2} - c_{1p_2} c_{2p_1}) g_2, g_3, \cdots, g_n\| \end{aligned}$$

By Lemma 2.1 and homogenity property of n-norm, we obtain

$$\begin{aligned}|c_{1p_1}|\mathcal{K} &= \text{abs}\left(\begin{vmatrix} c_{1p_1} & c_{2p_1} \\ c_{1p_2} & c_{2p_2}\end{vmatrix}\right)\|c_{1p_1}g_1 + c_{2p_1}g_2, g_2, g_3, \cdots, g_n\| \\ &= |c_{1p_1}|\text{abs}\left(\begin{vmatrix} c_{1p_1} & c_{2p_1} \\ c_{1p_2} & c_{2p_2}\end{vmatrix}\right)\|g_1, g_2, g_3, \cdots, g_n\|.\end{aligned}$$

Two sides are divided by $|c_{1p_1}|$, so $\mathcal{K} = \text{abs}\left(\begin{vmatrix} c_{1p_1} & c_{2p_1} \\ c_{1p_2} & c_{2p_2}\end{vmatrix}\right)\|g_1, g_2, g_3, \cdots, g_n\|$.

We conclude $\|p_1, p_2, g_3, \cdots, g_n\| = \text{abs}\left(\begin{vmatrix} c_{1p_1} & c_{2p_1} \\ c_{1p_2} & c_{2p_2}\end{vmatrix}\right)\|g_1, g_2, g_3, \cdots, g_n\|$. ■

The coefficients from p_1 and p_2 can be taken out of the n-norm and become a constant. The constant is obtained from the absolute value of the matrix determinant of the coefficients. Now we recall that the determinant has property

$$\begin{vmatrix} c_{1p_1}+b_1 & c_{2p_1} & \cdots & c_{2p_1} \\ c_{1p_2}+b_2 & c_{2p_2} & \cdots & c_{2p_2} \\ \vdots & \vdots & \ddots & \vdots \\ c_{1p_n}+b_n & c_{2p_n} & \cdots & c_{2p_n}\end{vmatrix} = \begin{vmatrix} c_{1p_1} & c_{2p_1} & \cdots & c_{2p_1} \\ c_{1p_2} & c_{2p_2} & \cdots & c_{2p_2} \\ \vdots & \vdots & \ddots & \vdots \\ c_{1p_n} & c_{2p_n} & \cdots & c_{2p_n}\end{vmatrix} + \begin{vmatrix} b_1 & c_{2p_1} & \cdots & c_{2p_1} \\ b_2 & c_{2p_2} & \cdots & c_{2p_2} \\ \vdots & \vdots & \ddots & \vdots \\ b_n & c_{2p_n} & \cdots & c_{2p_n}\end{vmatrix}.$$

Next, we continue with $n = 3$. Since it is getting a bit complicated, we have to prepare it by presenting this lemma.

Lemma 2.3. *Let a real square matrix* $\mathcal{S}_3 := \begin{pmatrix} c_{1p_1} & c_{2p_1} & c_{3p_1} \\ c_{1p_2} & c_{2p_2} & c_{3p_2} \\ c_{1p_3} & c_{2p_3} & c_{3p_3}\end{pmatrix}$, *and*

$$A_2 := \begin{vmatrix} c_{1p_1} & c_{2p_1} \\ c_{1p_2} & c_{2p_2}\end{vmatrix}, \quad A_3 := \begin{vmatrix} c_{1p_1} & c_{3p_1} \\ c_{1p_2} & c_{3p_2}\end{vmatrix}, \quad B_2 := \begin{vmatrix} c_{1p_1} & c_{2p_1} \\ c_{1p_3} & c_{2p_3}\end{vmatrix}, \quad B_3 := \begin{vmatrix} c_{1p_1} & c_{3p_1} \\ c_{1p_3} & c_{3p_3}\end{vmatrix}.$$

We have $\begin{vmatrix} A_2 & A_3 \\ B_2 & B_3\end{vmatrix} = c_{1p_1}|\mathcal{S}_3|$.

Proof. Suppose that $\mathcal{S}_3$ and A_i, B_i with $i = 2, 3$ as above. Next, check that

$$\begin{vmatrix} A_2 & A_3 \\ B_2 & B_3\end{vmatrix} = \begin{vmatrix} (c_{1p_1}c_{2p_2} - c_{2p_1}c_{1p_2}) & A_3 \\ (c_{1p_1}c_{2p_3} - c_{2p_1}c_{1p_3}) & B_3\end{vmatrix} = c_{1p_1}\begin{vmatrix} c_{2p_2} & A_3 \\ c_{2p_3} & B_3\end{vmatrix} - c_{2p_1}\begin{vmatrix} c_{1p_2} & A_3 \\ c_{1p_3} & B_3\end{vmatrix}.$$

Meanwhile, we obtain

$$\begin{vmatrix} c_{2p_2} & A_3 \\ c_{2p_3} & B_3\end{vmatrix} = \begin{vmatrix} c_{2p_2} & (c_{1p_1}c_{3p_2} - c_{1p_2}c_{3p_1}) \\ c_{2p_3} & (c_{1p_1}c_{3p_3} - c_{1p_3}c_{3p_1})\end{vmatrix} = c_{1p_1}\begin{vmatrix} c_{2p_2} & c_{3p_2} \\ c_{2p_3} & c_{3p_3}\end{vmatrix} - c_{3p_1}\begin{vmatrix} c_{2p_2} & c_{1p_2} \\ c_{2p_3} & c_{1p_3}\end{vmatrix}$$

and

$$\begin{vmatrix} c_{1p_2} & A_3 \\ c_{1p_3} & B_3\end{vmatrix} = \begin{vmatrix} c_{1p_2} & (c_{1p_1}c_{3p_2} - c_{1p_2}c_{3p_1}) \\ c_{1p_3} & (c_{1p_1}c_{3p_3} - c_{1p_3}c_{3p_1})\end{vmatrix} = c_{1p_1}\begin{vmatrix} c_{1p_2} & c_{3p_2} \\ c_{1p_3} & c_{3p_3}\end{vmatrix} - c_{3p_1}\begin{vmatrix} c_{1p_2} & c_{1p_2} \\ c_{1p_3} & c_{1p_3}\end{vmatrix}.$$

Since $\begin{vmatrix} c_{1p_2} & c_{1p_2} \\ c_{1p_3} & c_{1p_3}\end{vmatrix} = 0$, then

$$\begin{aligned}\begin{vmatrix} A_2 & A_3 \\ B_2 & B_3\end{vmatrix} &= c_{1p_1}\left(c_{1p_1}\begin{vmatrix} c_{2p_2} & c_{3p_2} \\ c_{2p_3} & c_{3p_3}\end{vmatrix} - c_{2p_1}\begin{vmatrix} c_{1p_2} & c_{3p_2} \\ c_{1p_3} & c_{3p_3}\end{vmatrix} + c_{3p_1}\begin{vmatrix} c_{1p_2} & c_{2p_2} \\ c_{1p_3} & c_{2p_3}\end{vmatrix}\right) \\ &= c_{1p_1}|\mathcal{S}_3|.\end{aligned}$$

The proof is complete. ∎

The lemma above tells us how the determinant of the real square matrix $n = 3$ relates to the determinant of the real square matrix $n = 2$. It is very useful and facilitates us in proving the proposition below.

Proposition 2.4. *Let an n-normed space X, (2.1), (2.2), and $\mathcal{S}_3$ (see Lemma 2.3). If $p_1, p_2, p_3 \in Y$ with $p_1 = \sum_{i=1}^{n} c_{ip_1} g_i$, $p_2 = \sum_{i=1}^{n} c_{ip_2} g_i$, and $p_3 = \sum_{i=1}^{n} c_{ip_3} g_i$, then*

$$\|p_1, p_2, p_3, g_4, \cdots, g_n\| = abs(|\mathcal{S}_3|) \|g_1, g_2, g_3, g_4, \cdots, g_n\|.$$

Proof. With all of assumptions, we use Lemma 2.1 to get

$$\begin{aligned} &\|p_1, p_2, p_3, g_4, \cdots, g_n\| \\ &= \left\| \sum_{i=1}^{n} c_{ip_1} g_i, \sum_{i=1}^{n} c_{ip_2} g_i, \sum_{i=1}^{n} c_{ip_3} g_i, g_4, \cdots, g_n \right\| \\ &= \left\| \sum_{i=1}^{3} c_{ip_1} g_i, \sum_{i=1}^{3} c_{ip_2} g_i, \sum_{i=1}^{3} c_{ip_3} g_i, g_4, \cdots, g_n \right\|. \end{aligned} \tag{2.5}$$

First case: for $\begin{bmatrix} c_{1p_1} & c_{2p_1} & c_{3p_1} \end{bmatrix}$, $\begin{bmatrix} c_{1p_2} & c_{2p_2} & c_{3p_2} \end{bmatrix}$, and $\begin{bmatrix} c_{1p_3} & c_{2p_3} & c_{3p_3} \end{bmatrix}$ be linearly dependent in $\mathbb{R}^3$, we have a trivial case. Check that $|\mathcal{S}_3| = 0$ and

$$\left\| \sum_{i=1}^{3} c_{ip_1} g_i, \sum_{i=1}^{3} c_{ip_2} g_i, \sum_{i=1}^{3} c_{ip_3} g_i, g_4, \cdots, g_n \right\| = 0.$$

Thus, $\|p_1, p_2, p_3, g_4, \cdots, g_n\| = 0$ and $\text{abs}(|\mathcal{S}_3|) \|g_1, g_2, g_3, g_4, \cdots, g_n\| = 0$.

Second case: for $\begin{bmatrix} c_{1p_1} & c_{2p_1} & c_{3p_1} \end{bmatrix}$, $\begin{bmatrix} c_{1p_2} & c_{2p_2} & c_{3p_2} \end{bmatrix}$, and $\begin{bmatrix} c_{1p_3} & c_{2p_3} & c_{3p_3} \end{bmatrix}$ be linearly independent in $\mathbb{R}^3$. It can be checked that $|\mathcal{S}_3| \neq 0$ holds. We also have a linearly independent set $\left\{ \sum_{i=1}^{3} c_{ip_1} g_i, \sum_{i=1}^{3} c_{ip_2} g_i, \sum_{i=1}^{3} c_{ip_3} g_i, g_4, \cdots, g_n \right\}$, so

$$\left\| \sum_{i=1}^{3} c_{ip_1} g_i, \sum_{i=1}^{3} c_{ip_2} g_i, \sum_{i=1}^{3} c_{ip_3} g_i, g_4, \cdots, g_n \right\| \neq 0.$$

Here, without losing generality, let $c_{1p_1} \neq 0$ and define

$$\mathcal{K} := \left\| \sum_{i=1}^{3} c_{ip_1} g_i, \sum_{i=1}^{3} c_{ip_2} g_i, \sum_{i=1}^{3} c_{ip_3} g_i, g_4, \cdots, g_n \right\|.$$

Next, we have $|c_{1p_1}|^2 \mathcal{K} = \left\| \sum_{i=1}^{3} c_{ip_1} g_i, c_{1p_1} \sum_{i=1}^{3} c_{ip_2} g_i, c_{1p_1} \sum_{i=1}^{3} c_{ip_3} g_i, g_4, \cdots, g_n \right\|$ or

$$|c_{1p_1}|^2 \mathcal{K} = \left\| \sum_{i=1}^{3} c_{ip_1} g_i, c_{1p_2} \left(\sum_{i=1}^{3} c_{ip_1} g_i \right) + \sum_{i=2}^{3} A_i g_i, c_{1p_3} \left(\sum_{i=1}^{3} c_{ip_1} g_i \right) + \sum_{i=2}^{3} B_i g_i, g_4, \cdots, g_n \right\|.$$

where $A_2 = \begin{vmatrix} c_{1p_1} & c_{2p_1} \\ c_{1p_2} & c_{2p_2} \end{vmatrix}$, $A_3 = \begin{vmatrix} c_{1p_1} & c_{3p_1} \\ c_{1p_2} & c_{3p_2} \end{vmatrix}$, $B_2 = \begin{vmatrix} c_{1p_1} & c_{2p_1} \\ c_{1p_3} & c_{2p_3} \end{vmatrix}$, and $B_3 = \begin{vmatrix} c_{1p_1} & c_{3p_1} \\ c_{1p_3} & c_{3p_3} \end{vmatrix}$.

Lemma 2.1 and homogenity property of n-norm give us

$$|c_{1p_1}|^2\mathcal{K} = \left\|\sum_{i=1}^{3} c_{ip_1}g_i, \sum_{i=2}^{3} A_i g_i, \sum_{i=2}^{3} B_i g_i, g_4, \cdots, g_n\right\|$$

and then by Proposition 2.2, $|c_{1p_1}|\mathcal{K} = \text{abs}\left(\begin{vmatrix} A_2 & A_3 \\ B_2 & B_3 \end{vmatrix}\right)\|g_1, g_2, g_3, g_4, \cdots, g_n\|$ holds. Now, using Lemma 2.3, we get $|c_{1p_1}|\mathcal{K} = |c_{1p_1}|\text{abs}\left(|\mathcal{S}_3|\right)\|g_1, g_2, g_3, g_4, \cdots, g_n\|$.

We conclude that $\|p_1, p_2, p_3, g_4, \cdots, g_n\| = \text{abs}\left(|\mathcal{S}_3|\right)\|g_1, g_2, g_3, g_4, \cdots, g_n\|$. ■

We do not stop for the real square matrix $n = 3$, but we will work for a real square matrix $\mathcal{S}_n$ with $n \geq 3$. We need special matrices for the following lemma and theorem. Now, define a real square matrix $\mathcal{S}_n$ with $n \geq 3$,

$$\mathcal{S}_n := \begin{pmatrix} c_{1p_1} & c_{2p_1} & \cdots & c_{np_1} \\ c_{1p_2} & c_{2p_2} & \cdots & c_{np_2} \\ \vdots & \vdots & \ddots & \vdots \\ c_{1p_n} & c_{2p_n} & \cdots & c_{np_n} \end{pmatrix}. \tag{2.6}$$

We also define

$$\mathcal{T}_{(n-1)} := \begin{pmatrix} A_{2,2} & A_{2,3} & \cdots & A_{2,n} \\ A_{3,2} & A_{3,3} & \cdots & A_{3,n} \\ \vdots & \vdots & \ddots & \vdots \\ A_{n,2} & A_{n,3} & \cdots & A_{n,n} \end{pmatrix}, \tag{2.7}$$

where

$$\begin{array}{llll} A_{2,2} = \begin{vmatrix} c_{1p_1} & c_{2p_1} \\ c_{1p_2} & c_{2p_2} \end{vmatrix}, & A_{2,3} = \begin{vmatrix} c_{1p_1} & c_{3p_1} \\ c_{1p_2} & c_{3p_2} \end{vmatrix}, & \cdots, & A_{2,n} = \begin{vmatrix} c_{1p_1} & c_{np_1} \\ c_{1p_2} & c_{np_2} \end{vmatrix} \\ A_{3,2} = \begin{vmatrix} c_{1p_1} & c_{2p_1} \\ c_{1p_3} & c_{2p_3} \end{vmatrix}, & A_{3,3} = \begin{vmatrix} c_{1p_1} & c_{3p_1} \\ c_{1p_3} & c_{3p_3} \end{vmatrix}, & \cdots, & A_{3,n} = \begin{vmatrix} c_{1p_1} & c_{np_1} \\ c_{1p_3} & c_{np_3} \end{vmatrix} \\ \vdots & \vdots & \ddots & \vdots \\ A_{n,2} = \begin{vmatrix} c_{1p_1} & c_{2p_1} \\ c_{1p_n} & c_{2p_n} \end{vmatrix}, & A_{n,3} = \begin{vmatrix} c_{1p_1} & c_{3p_1} \\ c_{1p_n} & c_{3p_n} \end{vmatrix}, & \cdots, & A_{n,n} = \begin{vmatrix} c_{1p_1} & c_{np_1} \\ c_{1p_n} & c_{np_n} \end{vmatrix}. \end{array}$$

Here, we also give a property of determinant of a real square matrix, that is Laplace's expansion

$$\begin{vmatrix} c_{1p_1} & c_{2p_1} & \cdots & c_{np_1} \\ c_{1p_2} & c_{2p_2} & \cdots & c_{np_2} \\ \vdots & \vdots & \ddots & \vdots \\ c_{1p_n} & c_{2p_n} & \cdots & c_{np_n} \end{vmatrix} = \sum_{k=1}^{n}(-1)^{(k+1)}\, c_{kp_1}\left|\left[c_{ip_j}\right]_{i\neq k, i=1\ ,\cdots,n, j=2\ ,\cdots,n}\right|.$$

Lemma 2.5. *Let $\mathcal{S}_n$ and $\mathcal{T}_{(n-1)}$ (see (2.6) and (2.7)). We have*

$$\left|\mathcal{T}_{(n-1)}\right| = \left(c_{1p_1}\right)^{(n-2)}|\mathcal{S}_n|.$$

Proof. Suppose that $\mathcal{S}_n$ and $\mathcal{T}_{(n-1)}$ as above. Here, we need $(n-1)$ steps.

Step-1: We check that

$$\left|\mathcal{T}_{(n-1)}\right| = \begin{vmatrix} (c_{1p_1}c_{2p_2} - c_{1p_2}c_{2p_1}) & A_{2,3} & \cdots & A_{2,n} \\ (c_{1p_1}c_{2p_3} - c_{1p_3}c_{2p_1}) & A_{3,3} & \cdots & A_{3,n} \\ \vdots & \vdots & \ddots & \vdots \\ (c_{1p_1}c_{2p_n} - c_{1p_n}c_{2p_1}) & A_{n,3} & \cdots & A_{n,n} \end{vmatrix} = c_{1p_1}U_1 - c_{2p_1}V_1,$$

where $U_1 = \begin{vmatrix} c_{2p_2} & A_{2,3} & \cdots & A_{2,n} \\ c_{2p_3} & A_{3,3} & \cdots & A_{3,n} \\ \vdots & \vdots & \ddots & \vdots \\ c_{2p_n} & A_{n,3} & \cdots & A_{n,n} \end{vmatrix}$ and $V_1 = \begin{vmatrix} c_{1p_2} & A_{2,3} & \cdots & A_{2,n} \\ c_{1p_3} & A_{3,3} & \cdots & A_{3,n} \\ \vdots & \vdots & \ddots & \vdots \\ c_{1p_n} & A_{n,3} & \cdots & A_{n,n} \end{vmatrix}$. Meanwhile, using the property of determinant of a square of matrices, we obtain

$$\begin{aligned} V_1 &= \begin{vmatrix} c_{1p_2} & (c_{1p_1}c_{3p_2} - c_{1p_2}c_{3p_1}) & \cdots & (c_{1p_1}c_{np_2} - c_{1p_2}c_{np_1}) \\ c_{1p_3} & (c_{1p_1}c_{3p_3} - c_{1p_3}c_{3p_1}) & \cdots & (c_{1p_1}c_{np_3} - c_{1p_3}c_{np_1}) \\ \vdots & \vdots & \ddots & \vdots \\ c_{1p_n} & (c_{1p_1}c_{3p_n} - c_{1p_n}c_{3p_1}) & \cdots & (c_{1p_1}c_{np_n} - c_{1p_n}c_{np_1}) \end{vmatrix} \\ &= (c_{1p_1})^{(n-2)} \begin{vmatrix} c_{1p_2} & c_{3p_2} & \cdots & c_{np_2} \\ c_{1p_3} & c_{3p_3} & \cdots & c_{np_3} \\ \vdots & \vdots & \ddots & \vdots \\ c_{1p_n} & c_{3p_n} & \cdots & c_{np_n} \end{vmatrix}. \end{aligned}$$

Step-2: We check that

$$\begin{aligned} U_1 &= \begin{vmatrix} c_{2p_2} & (c_{1p_1}c_{3p_2} - c_{1p_2}c_{3p_1}) & \cdots & (c_{1p_1}c_{np_2} - c_{1p_2}c_{np_1}) \\ c_{2p_3} & (c_{1p_1}c_{3p_3} - c_{1p_3}c_{3p_1}) & \cdots & (c_{1p_1}c_{np_3} - c_{1p_3}c_{np_1}) \\ \vdots & \vdots & \ddots & \vdots \\ c_{2p_n} & (c_{1p_1}c_{3p_n} - c_{1p_n}c_{3p_1}) & \cdots & (c_{1p_1}c_{np_n} - c_{1p_n}c_{np_1}) \end{vmatrix} \\ &= c_{1p_1}U_2 - c_{3p_1}V_2, \end{aligned}$$

where $U_2 = \begin{vmatrix} c_{2p_2} & c_{3p_2} & \cdots & A_{2,n} \\ c_{2p_3} & c_{3p_3} & \cdots & A_{3,n} \\ \vdots & \vdots & \ddots & \vdots \\ c_{2p_n} & c_{3p_n} & \cdots & A_{n,n} \end{vmatrix}$ and $V_2 = \begin{vmatrix} c_{2p_2} & c_{1p_2} & \cdots & A_{2,n} \\ c_{2p_3} & c_{1p_3} & \cdots & A_{3,n} \\ \vdots & \vdots & \ddots & \vdots \\ c_{2p_n} & c_{1p_n} & \cdots & A_{n,n} \end{vmatrix}$. Meanwhile, using the property of determinant of a square of matrices, we obtain

$$\begin{aligned} V_2 &= \begin{vmatrix} c_{2p_2} & c_{1p_2} & \cdots & (c_{1p_1}c_{np_2} - c_{1p_2}c_{np_1}) \\ c_{2p_3} & c_{1p_3} & \cdots & (c_{1p_1}c_{np_3} - c_{1p_3}c_{np_1}) \\ \vdots & \vdots & \ddots & \vdots \\ c_{2p_n} & c_{1p_n} & \cdots & (c_{1p_1}c_{np_n} - c_{1p_n}c_{np_1}) \end{vmatrix} \\ &= (c_{1p_1})^{(n-3)} \begin{vmatrix} c_{1p_2} & c_{3p_2} & \cdots & c_{np_2} \\ c_{1p_3} & c_{3p_3} & \cdots & c_{np_3} \\ \vdots & \vdots & \ddots & \vdots \\ c_{1p_n} & c_{3p_n} & \cdots & c_{np_n} \end{vmatrix}. \end{aligned}$$

We have to go to Step-3, Step-4, $\cdots$, until Step-$(n-1)$: We check that

$$\begin{aligned}U_{(n-2)}=&\begin{vmatrix} c_{2p_2} & c_{3p_2} & \cdots & (c_{1p_1}c_{(n-1)p_2}-c_{1p_2}c_{(n-1)p_1}) & (c_{1p_1}c_{np_2}-c_{1p_2}c_{np_1}) \\ c_{2p_3} & c_{3p_3} & \cdots & (c_{1p_1}c_{(n-1)p_3}-c_{1p_3}c_{(n-1)p_1}) & (c_{1p_1}c_{np_3}-c_{1p_3}c_{np_1}) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ c_{2p_n} & c_{3p_n} & \cdots & (c_{1p_1}c_{(n-1)p_n}-c_{1p_n}c_{(n-1)p_1}) & (c_{1p_1}c_{np_n}-c_{1p_n}c_{np_1}) \end{vmatrix}\\ =&(c_{1p_1})^2U_{(n-1)}-c_{1p_1}c_{np_1}V_{(n-1)},\end{aligned}$$

where

$$U_{(n-1)}=\begin{vmatrix} c_{2p_2} & c_{3p_2} & \cdots & c_{(n-1)p_2} & c_{np_2} \\ c_{2p_3} & c_{3p_3} & \cdots & c_{(n-1)p_3} & c_{np_3} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ c_{2p_n} & c_{3p_n} & \cdots & c_{(n-1)p_n} & c_{np_n} \end{vmatrix} \text{ and } V_{(n-1)}=\begin{vmatrix} c_{2p_2} & c_{3p_2} & \cdots & c_{1p_2} & c_{np_2} \\ c_{2p_3} & c_{3p_3} & \cdots & c_{1p_3} & c_{np_3} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ c_{2p_n} & c_{3p_n} & \cdots & c_{(1p_n} & c_{np_n} \end{vmatrix}.$$

We see again $\left|\mathcal{T}_{(n-1)}\right|$. Using Step-1 until Step-$(n-1)$, we have

$$\begin{aligned}\left|\mathcal{T}_{(n-1)}\right| =&(c_{1p_1})^{(n-1)}U_{(n-1)}-(c_{1p_1})^{(n-2)}c_{np_1}V_{(n-1)}\\ &-(c_{1p_1})^{(n-2)}c_{(n-1)p_1}V_{(n-2)}-(c_{1p_1})^{(n-2)}c_{(n-2)p_1}V_{(n-3)}\\ &-\cdots-(c_{1p_1})^{(n-2)}c_{4p_1}V_3-(c_{1p_1})^{(n-2)}c_{3p_1}V_2-(c_{1p_1})^{(n-2)}c_{2p_1}V_1.\end{aligned}$$

Finally, we arrange to get

$$\left|\mathcal{T}_{(n-1)}\right| = (c_{1p_1})^{(n-2)}\sum_{k=1}^{n}(-1)^{(k+1)}\, c_{kp_1}|W_k| = (c_{1p_1})^{(n-2)}\,|\mathcal{S}_n|$$

where $W_k=\left[c_{ip_j}\right]_{i\neq k,i=1,\cdots,n,j=2,\cdots,n}$ and $k=1,\cdots,n$. The proof is complete. ■

Theorem 2.6. *Let an n-normed space X where $n\geq 3$, (2.1) and (2.2). Let also $\mathcal{S}_{(n-1)}$, $\mathcal{S}_n$, and $p_1,p_2,\cdots,p_n\in Y$ with*

$$p_1=\sum_{i=1}^{n}c_{ip_1}g_i,\quad p_2=\sum_{i=1}^{n}c_{ip_2}g_i,\quad \cdots,\quad \textit{and}\quad p_n=\sum_{i=1}^{n}c_{ip_n}g_i.$$

If $\|p_1,p_2,\cdots,p_{(n-1)},g_n\|=abs\left(\left|\mathcal{S}_{(n-1)}\right|\right)\|g_1,g_2,\cdots,g_{(n-1)},g_n\|$, then

$$\|p_1,p_2,\cdots,p_n\|=abs\left(|\mathcal{S}_n|\right)\|g_1,g_2,\cdots,g_n\|.$$

Proof. Now, take all of the assumptions. We write the following

$$\|p_1,p_2,\cdots,p_n\|=\left\|\sum_{i=1}^{n}c_{ip_1}g_i,\sum_{i=1}^{n}c_{ip_2}g_i,\cdots,\sum_{i=1}^{n}c_{ip_n}g_i\right\|. \tag{2.8}$$

First case: for n linearly dependent vectors

$$\begin{bmatrix}c_{1p_1} & c_{2p_1} & \cdots & c_{np_1}\end{bmatrix},\begin{bmatrix}c_{1p_2} & c_{2p_2} & \cdots & c_{np_2}\end{bmatrix},\cdots,\text{ and }\begin{bmatrix}c_{1p_n} & c_{2p_n} & \cdots & c_{np_n}\end{bmatrix},$$

it is easy to check trivial case, that is $|\mathcal{S}_n|=0$ and

$$\left\|\sum_{i=1}^{n}c_{ip_1}g_i,\sum_{i=1}^{n}c_{ip_2}g_i,\cdots,\sum_{i=1}^{n}c_{ip_n}g_i\right\|=0.$$

We get $\|p_1,p_2,\cdots,p_3\|=0$ and abs$(|\mathcal{S}_n|)\,\|g_1,g_2,\cdots,g_n\|=0$.

Second case: for n linearly independent vectors

$\begin{bmatrix} c_{1p_1} & c_{2p_1} & \cdots & c_{np_1} \end{bmatrix}, \begin{bmatrix} c_{1p_2} & c_{2p_2} & \cdots & c_{np_2} \end{bmatrix}, \cdots$, and $\begin{bmatrix} c_{1p_n} & c_{2p_n} & \cdots & c_{np_n} \end{bmatrix}$,

we can check that $p_1, p_2, \cdots, p_n$ are also linearly independent. Consequently, we obtain $|\mathcal{S}_n| \neq 0$ and $\|p_1, p_2, \cdots, p_n\| = \left\| \sum_{i=1}^{n} c_{ip_1} g_i, \sum_{i=1}^{n} c_{ip_2} g_i, \cdots, \sum_{i=1}^{n} c_{ip_n} g_i \right\| \neq 0$. Here, without losing generality, let $c_{1p_1} \neq 0$ and

$$\mathcal{K} := \left\| \sum_{i=1}^{n} c_{ip_1} g_i, \sum_{i=1}^{n} c_{ip_2} g_i, \cdots, \sum_{i=1}^{n} c_{ip_n} g_i \right\|.$$

Next, write $|c_{1p_1}|^{(n-1)} \mathcal{K} = \left\| \sum_{i=1}^{n} c_{ip_1} g_i, c_{1p_1} \left(\sum_{i=1}^{n} c_{ip_2} g_i \right), \cdots, c_{1p_1} \left(\sum_{i=1}^{n} c_{ip_n} g_i \right) \right\|$ or

$$|c_{1p_1}|^{(n-1)} \mathcal{K} = \left\| \sum_{i=1}^{n} c_{ip_1} g_i, c_{1p_2} \left(\sum_{i=1}^{n} c_{ip_1} g_i \right) + \sum_{i=2}^{n} A_{2,i} g_i, \cdots, c_{1p_n} \left(\sum_{i=1}^{n} c_{ip_1} g_i \right) + \sum_{i=2}^{n} A_{n,i} g_i \right\|.$$

where

$$\begin{array}{cccc}
A_{2,2} = \begin{vmatrix} c_{1p_1} & c_{2p_1} \\ c_{1p_2} & c_{2p_2} \end{vmatrix}, & A_{2,3} = \begin{vmatrix} c_{1p_1} & c_{3p_1} \\ c_{1p_2} & c_{3p_2} \end{vmatrix}, & \cdots, & A_{2,n} = \begin{vmatrix} c_{1p_1} & c_{np_1} \\ c_{1p_2} & c_{np_2} \end{vmatrix} \\
A_{3,2} = \begin{vmatrix} c_{1p_1} & c_{2p_1} \\ c_{1p_3} & c_{2p_3} \end{vmatrix}, & A_{3,3} = \begin{vmatrix} c_{1p_1} & c_{3p_1} \\ c_{1p_3} & c_{3p_3} \end{vmatrix}, & \cdots, & A_{3,n} = \begin{vmatrix} c_{1p_1} & c_{np_1} \\ c_{1p_3} & c_{np_3} \end{vmatrix} \\
\vdots & \vdots & \ddots & \vdots \\
A_{n,2} = \begin{vmatrix} c_{1p_1} & c_{2p_1} \\ c_{1p_n} & c_{2p_n} \end{vmatrix}, & A_{n,3} = \begin{vmatrix} c_{1p_1} & c_{3p_1} \\ c_{1p_n} & c_{3p_n} \end{vmatrix}, & \cdots, & A_{n,n} = \begin{vmatrix} c_{1p_1} & c_{np_1} \\ c_{1p_n} & c_{np_n} \end{vmatrix}.
\end{array}$$

By Lemma 2.1, $|c_{1p_1}|^{(n-1)} \mathcal{K} = \left\| \sum_{i=1}^{n} c_{ip_1} g_i, \sum_{i=2}^{n} A_{2,i} g_i, \cdots, \sum_{i=2}^{n} A_{n,i} g_i \right\|$ holds. By assumption, homogenity property of n-norm, and Lemma 2.1, we get

$$|c_{1p_1}|^{(n-2)} \mathcal{K} = \text{abs}\left(\left| \mathcal{T}_{(n-1)} \right| \right) \|g_1, g_2, \cdots, g_n\|.$$

Next, use Lemma 2.5 to show that $|c_{1p_1}|^{(n-2)} \mathcal{K} = |c_{1p_1}|^{(n-2)} \text{abs}(|\mathcal{S}_n|) \|g_1, g_2, \cdots, g_n\|$.

Hence, $\|p_1, p_2, \cdots, p_n\| = \text{abs}(|\mathcal{S}_n|) \|g_1, g_2, \cdots, g_n\|$. ∎

Here, Proposition 2.2, Proposition 2.4, and Theorem 2.6 form a pattern that becomes a proof technique of mathematical induction. It can be seen that if we work in Y and want to calculate the n-norm of n vectors in Y, then it is enough to take a product of the absolute value of a scalar with the n-norm of $g_1, g_2, \cdots, g_n \in Y$. The scalar is from the determinant of the real square matrix of coefficient (by n vectors in Y).

2.2. A Norm And An Inner Product On Y

We take in [11] that in general, for X be a real vector space, an inner product on X is a mapping $\langle \cdot, \cdot \rangle : X \times X \to \mathbb{R}$ such that satisfying

(1) $\langle x, x \rangle \geq 0$ for every $x \in X$; $\langle x, x \rangle = 0$ if and only if $x = 0 \in X$;
(2) $\langle x, y \rangle = \langle y, x \rangle$ for every $x, y \in X$;
(3) $\langle \alpha x, y \rangle = \alpha \langle x, y \rangle$ for every $x \in X$ and for every scalars $\alpha \in \mathbb{R}$;
(4) $\langle x_1 + x_2, y \rangle = \langle x_1, y \rangle + \langle x_2, y \rangle$, for every $x_1, x_2, y \in X$.

We say that a pair $(X, \langle \cdot, \cdot \rangle)$ is an inner product space. Meanwhile, a norm is a mapping $\| \cdot \| : X \to \mathbb{R}$ which satisfies

(1) $\|x\| \geq 0$, for every $x \in X$; $\|x\| = 0$ if and only if $x = 0 \in X$;
(2) $\|\alpha x\| = |\alpha| \, \|x\|$, for every $x \in X$ and for every scalar $\alpha \in \mathbb{R}$;
(3) $\|x + y\| \leq \|x\| + \|y\|$ for every $x, y \in X$.

Then, a pair of $(X, \| \cdot \|)$ is called a normed space.

Let us consider a real square matrix $M_n = \begin{bmatrix} c_{1p_1} & c_{2p_1} & \cdots & c_{np_1} \\ c_{1p_2} & c_{2p_2} & \cdots & c_{np_2} \\ \vdots & \vdots & \ddots & \vdots \\ c_{1p_n} & c_{2p_n} & \cdots & c_{np_n} \end{bmatrix}$. The determinant of a square matrix has property $|M_n|^2 = |M_n M_n^T|$ with M_n^T is the transpose of M_n. Note that

$$M_n M_n^T = \begin{bmatrix} \langle c_{p_1}, c_{p_1} \rangle_{\mathbb{R}^n} & \langle c_{p_1}, c_{p_2} \rangle_{\mathbb{R}^n} & \cdots & \langle c_{p_1}, c_{p_n} \rangle_{\mathbb{R}^n} \\ \langle c_{p_2}, c_{p_1} \rangle_{\mathbb{R}^n} & \langle c_{p_2}, c_{p_2} \rangle_{\mathbb{R}^n} & \cdots & \langle c_{p_2}, c_{p_n} \rangle_{\mathbb{R}^n} \\ \vdots & \vdots & \ddots & \vdots \\ \langle c_{p_n}, c_{p_1} \rangle_{\mathbb{R}^n} & \langle c_{p_n}, c_{p_2} \rangle_{\mathbb{R}^n} & \cdots & \langle c_{p_n}, c_{p_n} \rangle_{\mathbb{R}^n} \end{bmatrix},$$

where

$$\langle c_{p_i}, c_{p_j} \rangle_{\mathbb{R}^n} := \sum_{k=1}^{n} c_{kp_i} c_{kp_j}, \tag{2.9}$$

with $c_{p_i}, c_{p_j} \in \mathbb{R}^n$. By (2.9), we also have

$$\|c_p\|_{\mathbb{R}^n} := \sqrt{\langle c_p, c_p \rangle_{\mathbb{R}^n}}, \tag{2.10}$$

with $c_p \in \mathbb{R}^n$. It is easy to show that (2.9) is an inner product and (2.10) is a norm on $\mathbb{R}^n$. The readers can definitely do it. We see again that $\sqrt{|M_n M_n^T|}$ satisfies the properties of n-norm, so we have

$$\|c_{p_1}, c_{p_2}, \cdots, c_{p_n}\|_{\mathbb{R}^n} := \sqrt{|M_n M_n^T|}$$

where $c_{p_1}, c_{p_2}, \cdots, c_{p_n} \in \mathbb{R}^n$.

On $\mathbb{R}^n$, inspired by [1, 6], we define a norm with respect to G on Y as follows

$$\|p\|_G := \sqrt{\sum_{\{i_2, \cdots, i_n\} \subseteq \{1, \cdots, n\}} \|p, g_{i_2}, \cdots, g_{i_n}\|^2}$$

for every $p \in Y$. Without losing generality, we give $p_1 = p = \sum_{j=1}^{n} c_{jp} g_j$ and $p_k = g_k$ where $k = 2, 3, \cdots, n$. One may check that $p_k = g_k = \sum_{j=1}^{n} c_{jp_k} g_j$ where

$$c_{g_k} = \begin{bmatrix} c_{1g_k} & c_{2g_k} & \cdots & c_{ng_k} \end{bmatrix}$$

with $c_{ig_k} = 0$ while $i \neq k$ and $c_{ig_k} = 1$ while $i = k$. Next, we compute $\langle c_p, c_{g_k}\rangle_{\mathbb{R}^n} = c_{kp}$, $\langle c_{g_j}, c_{g_k}\rangle_{\mathbb{R}^n} = 0$ while $j \neq k$, and $\langle c_{g_j}, c_{g_k}\rangle_{\mathbb{R}^n} = 1$ while $j = k$. Now, we obtain

$$\left(M_n M_n^T\right)_{\{2,\cdots,n\}} = \begin{bmatrix} \langle c_p, c_p\rangle_{\mathbb{R}^n} & c_{2p} & \cdots & c_{np} \\ c_{2p} & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ c_{np} & 0 & \cdots & 1 \end{bmatrix} = \sum_{i=1}^{n} c_{ip}^2 - \sum_{i=2}^{n} c_{ip}^2 = c_{1p}^2$$

and $\|p, g_2, \cdots, g_n\|^2 = \left(M_n M_n^T\right)_{\{2,\cdots,n\}} \|g_1, g_2, \cdots, g_n\|^2 = c_{1p}^2 \|g_1, g_2, \cdots, g_n\|^2$. This result is equivalent to Lemma 2.1 and (2.3). Next, $\{2, \cdots, n\}$ can be replaced by $\{i_2, \cdots, i_n\} \subseteq \{1, \cdots, n\}$, so

$$\left(M_n M_n^T\right)_{\{i_2,\cdots,i_n\}\subseteq\{1,\cdots,n\}} = \begin{bmatrix} \langle c_p, c_p\rangle_{\mathbb{R}^n} & c_{i_2p} & \cdots & c_{i_np} \\ c_{i_2p} & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ c_{i_np} & 0 & \cdots & 1 \end{bmatrix} = c_{i_1p}^2.$$

Consequently, we get

$$\begin{aligned} \|p\|_G^2 &= \sum_{\{i_2,\cdots,i_n\}\subseteq\{1,\cdots,n\}} \|p, g_{i_2}, \cdots, g_{i_n}\|^2 \\ &= \left(\sum_{i=1}^{n} c_{ip}^2\right) \|g_1, g_2, \cdots, g_n\|^2 \\ &= \|c_p\|_{\mathbb{R}^n}^2 \|g_1, g_2, \cdots, g_n\|^2 \end{aligned}$$

for every $p \in Y$. Since $\|\cdot\|_{\mathbb{R}^n}$ is induced from $\langle\cdot,\cdot\rangle_{\mathbb{R}^n}$, then we get an inner product with respect to G on Y

$$\langle p_1, p_2\rangle_G := \langle c_{p_1}, c_{p_2}\rangle_{\mathbb{R}^n} \|g_1, g_2, \cdots, g_n\|^2,$$

for every $p_1, p_2 \in Y$.

In addition, we can replace $\{g_1, \cdots, g_n\}$ with a set of linearly independent vectors $\{f_1, \cdots, f_n\} \subseteq Y$ to define another norm and another inner product on Y. It can be obtained, although with more complicated steps.

3. Concluding remarks

We have investigated Y as a subspace of $(X, \|\cdot, \cdots, \cdot\|)$. We obtain that $(Y, \langle\cdot,\cdot\rangle_G)$ is an inner product space and $(Y, \|\cdot\|_G)$ is a normed space. On $(Y, \langle\cdot,\cdot\rangle_G)$, there are still several functionals that can be defined. One may check and follow in [2, 5, 7, 8, 11–13]. In particular, to define the m-inner product, we have to use $m < n$.

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References

[1] S. Ekariani, H. Gunawan, M. Idris, A contractive mapping theorem on the n-normed space of p-summable sequences, *J. Math. Analysis*, **4(1)** (2013) 1–7.

[2] C. R. Diminnie, S. Gähler, A. White, 2-inner product spaces, *Demonstratio Math.* **6** (1973) 525–536.

[3] S. Gähler, Lineare 2-normietre Räume, *Math. Nachr.* **28** (1965), 1–43.

[4] S. Gähler, Uber 2-Banach Räume, *Math. Nachr.* **42** (1969), 335–347.

[5] H. Gunawan, An inner product that makes a set of vectors orthonormal, *Austral. Math. Soc. Gaz.* **28** (2001) 194–197.

[6] H. Gunawan, The space of p-summable sequences and its natural n-norms, *Bull. Austral. Math. Soc.* **64** (2001), 137–147.

[7] H. Gunawan, On n-inner products, n-norms, and the Cauchy-Schwarz inequality, *Sci. Math. Jpn.* **55** (2002) 53–60.

[8] H. Gunawan, O. Neswan, W. Setya-Budhi, A formula for angles between two subspaces of inner product spaces, *Beitr. Algebra Geom.* **46** (2005) 311–329.

[9] S. Konca, M. Idris, H. Gunawan, p-summable sequence spaces with inner products, *Beu J. Sci. Techn.* **5** (1) (2015) 37–41.

[10] S. Konca, M. Idris, Equivalence among three 2-norms on the space of p-summable sequences, *Journal of Inequalities and Special Functions* **7**, 4 (2015) 218–224.

[11] E. Kreyszig, *Introductory Functional Analysis with Applications*, John Wiley & Sons, Inc., New York, 1978.

[12] A. Misiak, n-inner product spaces, *Math. Nachr.* **140** (1989) 299–319.

[13] A. Misiak, Orthogonality and orthonormality in n-inner product spaces, *Math. Nachr.* **143** (1989) 249–261.